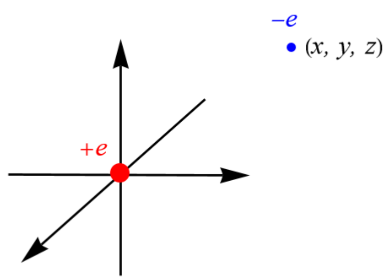
Problem 4.12

Work out the radial wave functions R_{30}, R_{31}, and R_{32}, using the recursion formula (Equation 4.76). Don't bother to normalize them.

Solution

The goal in this problem is to analyze an electrically neutral hydrogen atom, which has one proton and one electron. Because a proton is roughly 2000 times more massive than an electron, the proton's motion can be neglected to a good approximation. As such, let the proton lie at the origin of space. Ignore spin for now.



The electron is somewhere around the nucleus; solve the Schrödinger equation to determine its wave function.

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m_e}\nabla^2\Psi + V(x,y,z)\Psi(x,y,z,t)$$

The potential energy function $V(x,y,z)$ is determined from

$$\mathbf{F} = -\nabla V,$$

where $\mathbf{F}$ is given by Coulomb's law.

$$\frac{1}{4\pi\epsilon_0}\frac{(-e)(e)}{r^2}\,\hat{\mathbf{r}} = -\nabla V,$$

Since the force is only dependent on the spherical coordinate $r = \sqrt{x^2+y^2+z^2}$, the potential energy function is as well.

$$-\frac{1}{4\pi\epsilon_0}\frac{e^2}{r^2} = -\frac{dV}{dr}$$

Multiply both sides by -1 and then integrate both sides from ∞ to r.

$$\int_\infty^r \frac{1}{4\pi\epsilon_0}\frac{e^2}{r_0^2}\,dr_0 = \int_\infty^r \frac{dV}{dr}(r_0)\,dr_0$$

Evaluate the integrals.

$$-\frac{1}{4\pi\epsilon_0}\frac{e^2}{r_0}\bigg|_\infty^r = V(r) - \underbrace{V(\infty)}_{=0}$$

As a result,

$$V(r) = -\frac{e^2}{4\pi\epsilon_0}\frac{1}{r},$$

and Schrödinger's equation becomes

$$\begin{aligned} i\hbar\frac{\partial\Psi}{\partial t} &= -\frac{\hbar^2}{2m_e}\nabla^2\Psi + V(r)\Psi(r,\phi,\theta,t) \\ &= -\frac{\hbar^2}{2m_e}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\Psi}{\partial\phi^2}\right] + V(r)\Psi(r,\phi,\theta,t). \end{aligned}$$

The aim is to solve for $\Psi = \Psi(r,\theta,\phi,t)$ in all of space ($0 \le r < \infty$, $0 \le \theta \le \pi$, $0 \le \phi \le 2\pi$) for $t > 0$. Assuming a product solution of the form $\Psi(r,\theta,\phi,t) = R(r)\Theta(\theta)\xi(\phi)T(t)$ and plugging it into the PDE yields the following system of ODEs (see Problem 4.4).

$$\left.\begin{aligned} i\hbar\frac{T'(t)}{T(t)} &= E \\ \frac{1}{R(r)}\frac{d}{dr}\left(r^2R'(r)\right) - \frac{2m_er^2}{\hbar^2}[V(r)-E] &= F \\ \frac{\sin\theta}{\Theta(\theta)}\frac{d}{d\theta}\left(\Theta'(\theta)\sin\theta\right) + F\sin^2\theta &= G \\ -\frac{\xi''(\phi)}{\xi(\phi)} &= G \end{aligned}\right\}$$

The normalized products of angular eigenfunctions $\Theta(\theta)\xi(\phi)$ are called the spherical harmonics and are denoted by $Y_\ell^m(\theta,\phi)$. Solutions only exist if $F = \ell(\ell+1)$, where $\ell = 0, 1, 2, \ldots$, and if $G = m^2$ is an integer.

$$Y_\ell^m(\theta,\phi) = \sqrt{\frac{(2\ell+1)}{4\pi}\frac{(\ell-m)!}{(\ell+m)!}}\,e^{im\phi}P_\ell^m(\cos\theta), \qquad \begin{cases} \ell = 0,1,2,\ldots \\ m = -\ell,-\ell+1,\ldots,-1,0,1,\ldots,\ell-1,\ell \end{cases}$$

With these results the equation for $R(r)$ becomes

$$\frac{1}{R(r)}\frac{d}{dr}\left(r^2R'(r)\right) - \frac{2m_er^2}{\hbar^2}\left(-\frac{e^2}{4\pi\epsilon_0 r} - E\right) = \ell(\ell+1)$$

$$\frac{d}{dr}\left[r^2\frac{dR}{dr}(r)\right] + \left[2\left(\frac{m_ee^2}{4\pi\epsilon_0\hbar^2}\right)r + \frac{2m_er^2}{\hbar^2}E\right]R(r) - \ell(\ell+1)R(r) = 0. \tag{1}$$

Note that since we're interested in the bound states of the hydrogen atom (a proton and an electron paired together), $E < 0$. Also, the grouping of constants in parentheses is $1/a_0$, where a_0 is the Bohr radius. Make the change of variables,

$$s = \kappa r, \quad \text{where } \kappa = \frac{\sqrt{-8m_eE}}{\hbar}.$$

Consequently, equation (1) turns into

$$\frac{ds}{dr}\frac{d}{ds}\left[\left(\frac{s}{\kappa}\right)^2\frac{ds}{dr}\frac{d}{ds}R\left(\frac{s}{\kappa}\right)\right] + \left[\frac{2}{a_0}\left(\frac{s}{\kappa}\right) + \frac{2m_es^2}{\hbar^2\kappa^2}E\right]R\left(\frac{s}{\kappa}\right) - \ell(\ell+1)R\left(\frac{s}{\kappa}\right) = 0.$$

Use a new dependent variable,

$$w(s) = R\left(\frac{s}{\kappa}\right),$$

and simplify the left side.

$$\kappa\frac{d}{ds}\left[\left(\frac{s^2}{\kappa^2}\right)\kappa\frac{dw}{ds}\right]+\left[\frac{2}{a_0}\left(\frac{s}{\kappa}\right)+\frac{2m_e s^2}{\hbar^2}\left(-\frac{\hbar^2}{8m_e}\right)\right]w(s)-\ell(\ell+1)w(s)=0$$

$$\frac{d}{ds}\left(s^2\frac{dw}{ds}\right)+\left[\frac{2s}{a_0\kappa}-\frac{s^2}{4}-\ell(\ell+1)\right]w(s)=0$$

Make another change of variables.

$$w(s)=s^\ell e^{-s/2}u(s)$$

As a result,

$$\begin{aligned}
0&=\frac{d}{ds}\left\{s^2\frac{d}{ds}[s^\ell e^{-s/2}u(s)]\right\}+\left[\frac{2s}{a_0\kappa}-\frac{s^2}{4}-\ell(\ell+1)\right]s^\ell e^{-s/2}u(s)\\
&=\frac{d}{ds}\left\{s^2\left[\ell s^{\ell-1}e^{-s/2}u(s)+s^\ell\left(-\frac{1}{2}\right)e^{-s/2}u(s)+s^\ell e^{-s/2}\frac{du}{ds}\right]\right\}+\left[\frac{2s}{a_0\kappa}-\frac{s^2}{4}-\ell(\ell+1)\right]s^\ell e^{-s/2}u(s)\\
&=\frac{d}{ds}\left(\ell s^{\ell+1}e^{-s/2}u(s)-\frac{1}{2}s^{\ell+2}e^{-s/2}u(s)+s^{\ell+2}e^{-s/2}\frac{du}{ds}\right)+\left[\frac{2s}{a_0\kappa}-\frac{s^2}{4}-\ell(\ell+1)\right]s^\ell e^{-s/2}u(s)\\
&=\left[\cancel{\ell(\ell+1)s^\ell e^{-s/2}u(s)}+\ell s^{\ell+1}\left(-\frac{1}{2}\right)e^{-s/2}u(s)+\ell s^{\ell+1}e^{-s/2}\frac{du}{ds}\right]\\
&\qquad-\frac{1}{2}\left[(\ell+2)s^{\ell+1}e^{-s/2}u(s)+\cancel{s^{\ell+2}\left(-\frac{1}{2}\right)e^{-s/2}u(s)}+s^{\ell+2}e^{-s/2}\frac{du}{ds}\right]\\
&\qquad\quad+\left[(\ell+2)s^{\ell+1}e^{-s/2}\frac{du}{ds}+s^{\ell+2}\left(-\frac{1}{2}\right)e^{-s/2}\frac{du}{ds}+s^{\ell+2}e^{-s/2}\frac{d^2u}{ds^2}\right]\\
&\qquad\qquad+\frac{2}{a_0\kappa}s^{\ell+1}e^{-s/2}u(s)-\cancel{\frac{1}{4}s^{\ell+2}e^{-s/2}u(s)}-\cancel{\ell(\ell+1)s^\ell e^{-s/2}u(s)}\\
&=s^{\ell+2}e^{-s/2}\frac{d^2u}{ds^2}+(2\ell+2-s)s^{\ell+1}e^{-s/2}\frac{du}{ds}+\left(\frac{2}{a_0\kappa}-\ell-1\right)s^{\ell+1}e^{-s/2}u(s).
\end{aligned}$$

Multiply both sides by $e^{s/2}$.

$$s^{\ell+2}\frac{d^2u}{ds^2}+(2\ell+2-s)s^{\ell+1}\frac{du}{ds}+\left(\frac{2}{a_0\kappa}-\ell-1\right)s^{\ell+1}u(s)=0$$

Divide both sides by $s^{\ell+1}$.

$$s\frac{d^2u}{ds^2}+[(2\ell+1)+1-s]\frac{du}{ds}+\left(\frac{2}{a_0\kappa}-\ell-1\right)u(s)=0,\quad 0<s<\infty \tag{2}$$

This is the generalized Laguerre differential equation. Normalizable solutions exist only if the quantity in parentheses multiplying $u(s)$ is a nonnegative integer $(0,1,2,\ldots)$. It's this fact that allows us to determine the eigenenergies of the bound states of hydrogen. Let N be the nonnegative integer.

$$\frac{2}{a_0\kappa}-\ell-1=N\quad\rightarrow\quad\frac{2}{a_0\kappa}=N+\ell+1$$

The number on the right side is a positive integer $(1, 2, \ldots)$ and is denoted by n.

$$\frac{2}{a_0\kappa} = n \quad \rightarrow \quad 2\left(\frac{m_e e^2}{4\pi\epsilon_0\hbar^2}\right)\left(\frac{\hbar}{\sqrt{-8m_eE}}\right) = n$$

Solve for E.

$$\boxed{E_n = -\frac{m_e e^4}{32\pi^2\epsilon_0^2\hbar^2 n^2} = -\left[\frac{m_e}{2\hbar^2}\left(\frac{e^2}{4\pi\epsilon_0}\right)^2\right]\frac{1}{n^2} = \frac{E_1}{n^2}, \quad n = 1, 2, \ldots}$$

In the case that $\frac{2}{a_0\kappa} - \ell - 1$, or $n - \ell - 1$, is a nonnegative integer, equation (2) has a polynomial solution called the generalized Laguerre polynomial of degree $n - \ell - 1$ and order $2\ell + 1$.

$$\begin{aligned} u(s) = AL_{n-\ell-1}^{2\ell+1}(s) &= A[(2\ell+1) + (n-\ell-1)]! \sum_{j=0}^{n-\ell-1} \frac{(-1)^j}{j![(n-\ell-1)-j]![(2\ell+1)+j]!} s^j \\ &= A(n+\ell)! \sum_{j=0}^{n-\ell-1} \frac{(-1)^j}{j!(n-\ell-j-1)!(2\ell+j+1)!} s^j \end{aligned}$$

This means

$$\begin{aligned} w(s) &= s^\ell e^{-s/2} u(s) \\ &= As^\ell e^{-s/2} L_{n-\ell-1}^{2\ell+1}(s), \end{aligned}$$

and

$$\begin{aligned} R(r) &= A(\kappa r)^\ell e^{-(\kappa r)/2} L_{n-\ell-1}^{2\ell+1}(\kappa r) \\ &= A\left(\frac{\sqrt{-8m_eE}}{\hbar} r\right)^\ell \exp\left[-\frac{1}{2}\left(\frac{\sqrt{-8m_eE}}{\hbar} r\right)\right] L_{n-\ell-1}^{2\ell+1}\left(\frac{\sqrt{-8m_eE}}{\hbar} r\right) \\ &= A\left(\frac{m_e e^2}{2\pi\epsilon_0\hbar^2 n} r\right)^\ell \exp\left[-\frac{1}{2}\left(\frac{m_e e^2}{2\pi\epsilon_0\hbar^2 n} r\right)\right] L_{n-\ell-1}^{2\ell+1}\left(\frac{m_e e^2}{2\pi\epsilon_0\hbar^2 n} r\right) \\ &= A\left(\frac{2}{na_0} r\right)^\ell \exp\left(-\frac{r}{na_0}\right) L_{n-\ell-1}^{2\ell+1}\left(\frac{2}{na_0} r\right) \end{aligned} \tag{3}$$

since $w(s) = R(s/\kappa)$, or $w(\kappa r) = R(r)$. Now that E is known, the last of the four ODEs can be solved.

$$i\hbar\frac{T'(t)}{T(t)} = E \quad \Rightarrow \quad T(t) = e^{-iEt/\hbar}$$

The temporal eigenfunctions are then

$$\begin{aligned} T_n(t) &= \exp\left[-\frac{i}{\hbar}\left(-\frac{m_e e^4}{32\pi^2\epsilon_0^2\hbar^2 n^2}\right) t\right] \\ &= \exp\left(i\frac{m_e e^4}{32\pi^2\epsilon_0^2\hbar^3 n^2} t\right). \end{aligned}$$

Determine the arbitrary constant A by requiring the stationary states to be normalized.

$$
\begin{aligned}
1 = \iiint_{\text{all space}} |\Psi_{n\ell m}(r,\theta,\phi,t)|^2\,d\mathcal{V} &= \iiint_{\text{all space}} |R(r)\Theta(\theta)\xi(\phi)T(t)|^2\,d\mathcal{V} \\
&= \iiint_{\text{all space}} \left|R_{n\ell}(r)Y_\ell^m(\theta,\phi)e^{-iE_nt/\hbar}\right|^2 d\mathcal{V} \\
&= \int_0^\pi \int_0^{2\pi} \int_0^\infty |R_{n\ell}(r)|^2|Y_\ell^m(\theta,\phi)|^2(r^2\sin\theta\,dr\,d\phi\,d\theta) \\
&= \underbrace{\left[\int_0^\infty r^2|R_{n\ell}(r)|^2\,dr\right]}_{=1}\underbrace{\left[\int_0^\pi \int_0^{2\pi} |Y_\ell^m(\theta,\phi)|^2\sin\theta\,d\phi\,d\theta\right]}_{=1}
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
1 &= \int_0^\infty r^2|R_{n\ell}(r)|^2\,dr \\
&= \int_0^\infty r^2|A|^2\left(\frac{2}{na_0}r\right)^{2\ell}\exp\left(-\frac{2r}{na_0}\right)\left[L_{n-\ell-1}^{2\ell+1}\left(\frac{2}{na_0}r\right)\right]^2 dr.
\end{aligned}
$$

Make the following substitution.

$$
\begin{aligned}
\nu &= \frac{2}{na_0}r && \rightarrow && r = \frac{na_0}{2}\nu \\
d\nu &= \frac{2}{na_0}\,dr && \rightarrow && dr = \frac{na_0}{2}\,d\nu
\end{aligned}
$$

The integral then becomes

$$
\begin{aligned}
1 &= |A|^2\int_0^\infty \left(\frac{na_0}{2}\nu\right)^2\nu^{2\ell}e^{-\nu}\left[L_{n-\ell-1}^{2\ell+1}(\nu)\right]^2\left(\frac{na_0}{2}\,d\nu\right) \\
&= |A|^2\left(\frac{na_0}{2}\right)^3\left\{\int_0^\infty \nu^{2\ell+2}e^{-\nu}\left[L_{n-\ell-1}^{2\ell+1}(\nu)\right]^2 d\nu\right\} \\
&= |A|^2\left(\frac{na_0}{2}\right)^3\left\{\frac{[(n-\ell-1)+(2\ell+1)]!}{(n-\ell-1)!}[2(n-\ell-1)+(2\ell+1)+1]\right\} \\
&= |A|^2\left(\frac{na_0}{2}\right)^3\frac{(n+\ell)!}{(n-\ell-1)!}(2n).
\end{aligned}
$$

The evaluation of this integral is involved and will be elaborated on in Problem 4.63. Solve for A.

$$
A = \sqrt{\left(\frac{2}{na_0}\right)^3\frac{(n-\ell-1)!}{2n(n+\ell)!}}
$$

As a result, equation (3) becomes

$$R_{n\ell}(r) = A\left(\frac{2}{na_0}r\right)^{\ell} \exp\left(-\frac{r}{na_0}\right) L_{n-\ell-1}^{2\ell+1}\left(\frac{2}{na_0}r\right)$$

$$= \sqrt{\left(\frac{2}{na_0}\right)^3 \frac{(n-\ell-1)!}{2n(n+\ell)!}} \left(\frac{2}{na_0}r\right)^{\ell} \exp\left(-\frac{r}{na_0}\right) L_{n-\ell-1}^{2\ell+1}\left(\frac{2}{na_0}r\right)$$

$$= \sqrt{\left(\frac{2}{na_0}\right)^3 \frac{(n-\ell-1)!}{2n(n+\ell)!}} \left(\frac{2}{na_0}r\right)^{\ell} \exp\left(-\frac{r}{na_0}\right) \left[(n+\ell)! \sum_{j=0}^{n-\ell-1} \frac{(-1)^j}{j!(n-\ell-j-1)!(2\ell+j+1)!}\left(\frac{2}{na_0}r\right)^j\right].$$

The formula for the radial eigenfunctions is therefore

$$\boxed{R_{n\ell}(r) = \frac{2}{n^2}\sqrt{\frac{(n+\ell)!(n-\ell-1)!}{a_0^3}} \left[\sum_{j=0}^{n-\ell-1} \frac{(-1)^j}{j!(n-\ell-j-1)!(2\ell+j+1)!}\left(\frac{2}{na_0}r\right)^{j+\ell}\right] \exp\left(-\frac{r}{na_0}\right), \quad \begin{cases} n = 1, 2, 3, \ldots \\ \ell = 0, 1, 2, \ldots \\ n > \ell \end{cases}}$$

For example,

$$R_{30}(r) = \frac{2}{3^2}\sqrt{\frac{(3+0)!(3-0-1)!}{a_0^3}} \left[\sum_{j=0}^{3-0-1} \frac{(-1)^j}{j!(3-0-j-1)![2(0)+j+1]!}\left(\frac{2}{3a_0}r\right)^{j+0}\right] \exp\left(-\frac{r}{3a_0}\right)$$

$$= \frac{2}{9}\sqrt{\frac{12}{a_0^3}} \left[\sum_{j=0}^{2} \frac{(-1)^j}{j!(2-j)!(j+1)!}\left(\frac{2}{3a_0}r\right)^j\right] \exp\left(-\frac{r}{3a_0}\right)$$

$$= \frac{4}{9}\sqrt{\frac{3}{a_0^3}} \left[\frac{(-1)^0}{0!(2-0)!(0+1)!}\left(\frac{2}{3a_0}r\right)^0 + \frac{(-1)^1}{1!(2-1)!(1+1)!}\left(\frac{2}{3a_0}r\right)^1 + \frac{(-1)^2}{2!(2-2)!(2+1)!}\left(\frac{2}{3a_0}r\right)^2\right] \exp\left(-\frac{r}{3a_0}\right)$$

$$= \frac{4}{9}\sqrt{\frac{3}{a_0^3}} \left[\frac{1}{2} - \frac{1}{2}\left(\frac{2}{3a_0}r\right) + \frac{1}{12}\left(\frac{2}{3a_0}r\right)^2\right] \exp\left(-\frac{r}{3a_0}\right)$$

$$= \frac{2}{9}\sqrt{\frac{3}{a_0^3}} \left[1 - \frac{2}{3}\left(\frac{r}{a_0}\right) + \frac{2}{27}\left(\frac{r}{a_0}\right)^2\right] \exp\left(-\frac{r}{3a_0}\right)$$

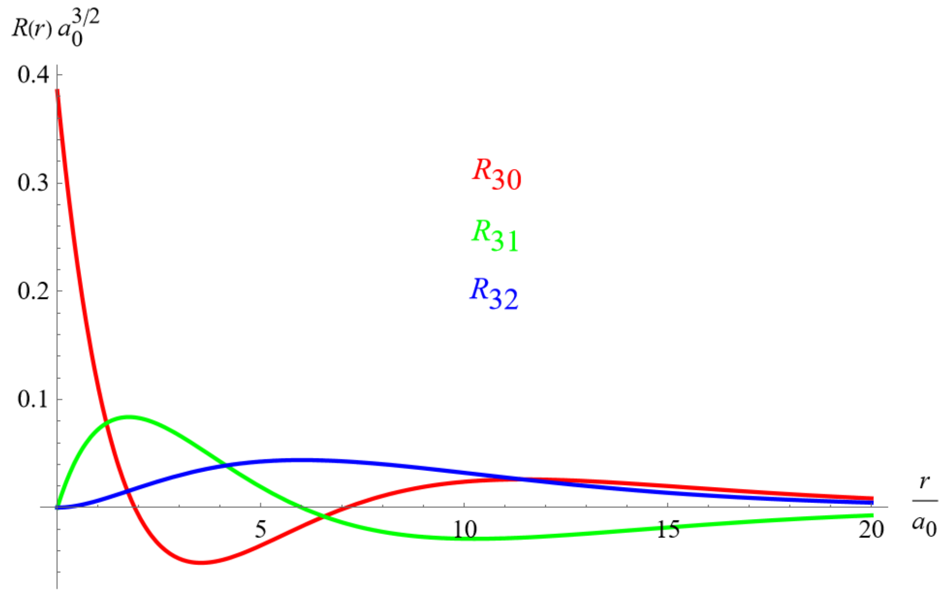
and

$$\begin{aligned}
R_{31}(r) &= \frac{2}{3^2}\sqrt{\frac{(3+1)!(3-1-1)!}{a_0^3}}\left[\sum_{j=0}^{3-1-1}\frac{(-1)^j}{j!(3-1-j-1)![2(1)+j+1]!}\left(\frac{2}{3a_0}r\right)^{j+1}\right]\exp\left(-\frac{r}{3a_0}\right) \\
&= \frac{2}{9}\sqrt{\frac{24}{a_0^3}}\left[\sum_{j=0}^{1}\frac{(-1)^j}{j!(1-j)!(j+3)!}\left(\frac{2}{3a_0}r\right)^{j+1}\right]\exp\left(-\frac{r}{3a_0}\right) \\
&= \frac{4}{9}\sqrt{\frac{6}{a_0^3}}\left[\frac{(-1)^0}{0!(1-0)!(0+3)!}\left(\frac{2}{3a_0}r\right)^{0+1}+\frac{(-1)^1}{1!(1-1)!(1+3)!}\left(\frac{2}{3a_0}r\right)^{1+1}\right]\exp\left(-\frac{r}{3a_0}\right) \\
&= \frac{4}{9}\sqrt{\frac{6}{a_0^3}}\left[\frac{1}{6}\left(\frac{2}{3a_0}r\right)-\frac{1}{24}\left(\frac{2}{3a_0}r\right)^2\right]\exp\left(-\frac{r}{3a_0}\right) \\
&= \frac{4}{81}\sqrt{\frac{6}{a_0^3}}\left(\frac{r}{a_0}\right)\left[1-\frac{1}{6}\left(\frac{r}{a_0}\right)\right]\exp\left(-\frac{r}{3a_0}\right)
\end{aligned}$$

and

$$\begin{aligned}
R_{32}(r) &= \frac{2}{3^2}\sqrt{\frac{(3+2)!(3-2-1)!}{a_0^3}}\left[\sum_{j=0}^{3-2-1}\frac{(-1)^j}{j!(3-2-j-1)![2(2)+j+1]!}\left(\frac{2}{3a_0}r\right)^{j+2}\right]\exp\left(-\frac{r}{3a_0}\right) \\
&= \frac{2}{9}\sqrt{\frac{120}{a_0^3}}\left[\sum_{j=0}^{0}\frac{(-1)^j}{j!(0-j)!(j+5)!}\left(\frac{2}{3a_0}r\right)^{j+2}\right]\exp\left(-\frac{r}{3a_0}\right) \\
&= \frac{2}{9}\sqrt{\frac{120}{a_0^3}}\left[\frac{(-1)^0}{0!(0-0)!(0+5)!}\left(\frac{2}{3a_0}r\right)^{0+2}\right]\exp\left(-\frac{r}{3a_0}\right) \\
&= \frac{2}{9}\sqrt{\frac{120}{a_0^3}}\left[\frac{1}{120}\left(\frac{2}{3a_0}r\right)^2\right]\exp\left(-\frac{r}{3a_0}\right) \\
&= \frac{1}{1215}\sqrt{\frac{120}{a_0^3}}\left(\frac{r}{a_0}\right)^2\exp\left(-\frac{r}{3a_0}\right).
\end{aligned}$$

Below are plots of these radial eigenfunctions.



To conclude, the stationary states of the hydrogen atom are

$$\Psi_{n\ell m}(r,\phi,\theta,t) = R_{n\ell}(r)Y_\ell^m(\theta,\phi)T_n(t)$$

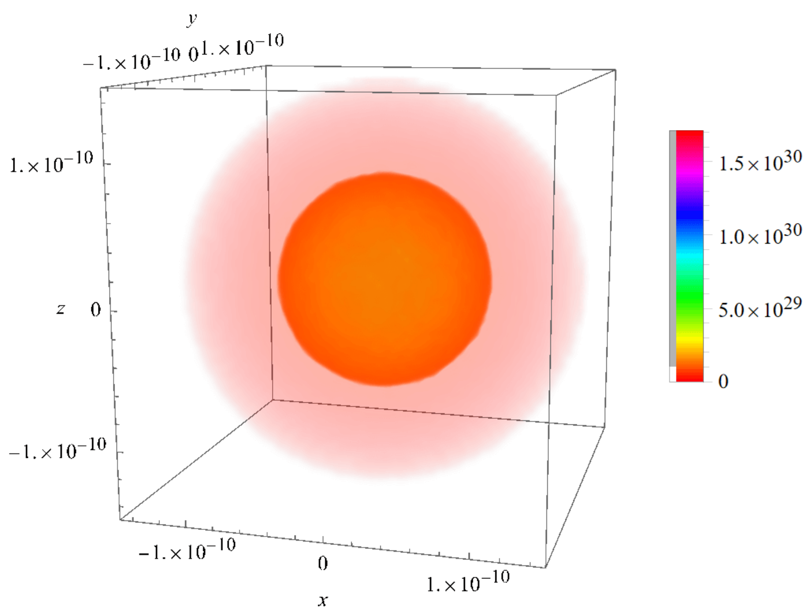
$$= \sqrt{\left(\frac{2}{na_0}\right)^3 \frac{(n-\ell-1)!}{2n(n+\ell)!}} \left(\frac{2}{na_0}r\right)^\ell \exp\left(-\frac{r}{na_0}\right) L_{n-\ell-1}^{2\ell+1}\left(\frac{2}{na_0}r\right) Y_\ell^m(\theta,\phi)e^{-iE_nt/\hbar}$$

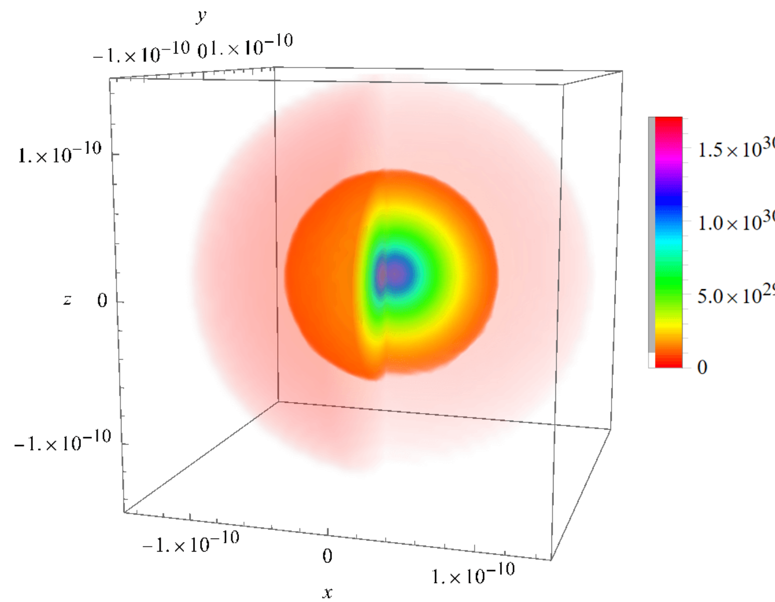
with $|m| \leq \ell < n \geq 1$, where

$$a_0 = \frac{4\pi\epsilon_0\hbar^2}{m_e e^2} \approx 0.529 \times 10^{-10} \text{ m} \quad \text{and} \quad E_n = -\left[\frac{m_e}{2\hbar^2}\left(\frac{e^2}{4\pi\epsilon_0}\right)^2\right]\frac{1}{n^2} \approx -\frac{2.18 \times 10^{-18} \text{ J}}{n^2}.$$

The probability density of the electron's position in each stationary state of hydrogen is visualized below (SI units).

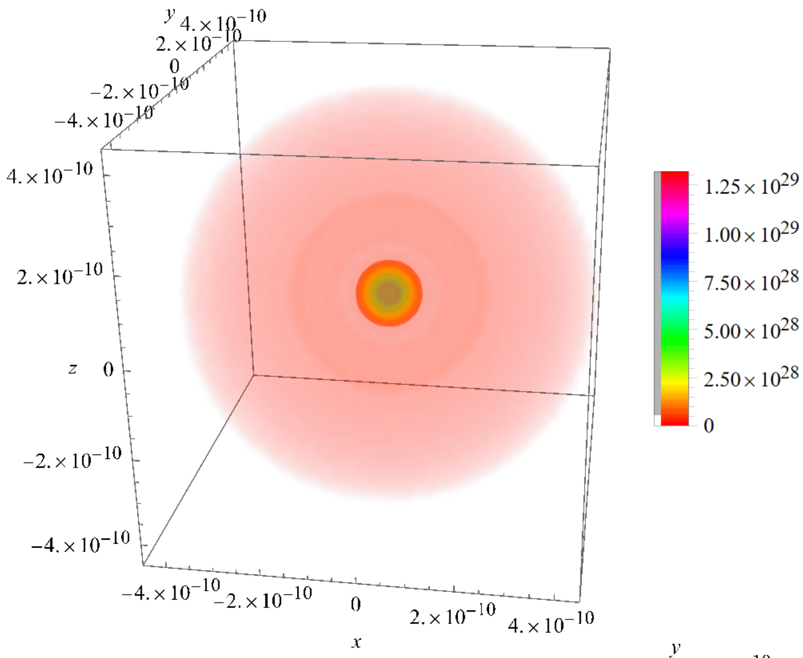
$$|\Psi_{100}(r, \theta, \phi, t)|^2$$

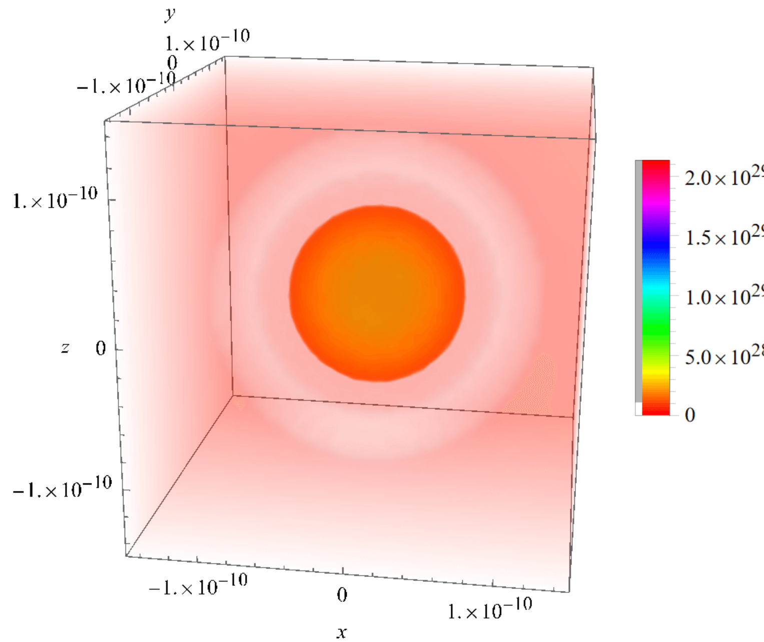


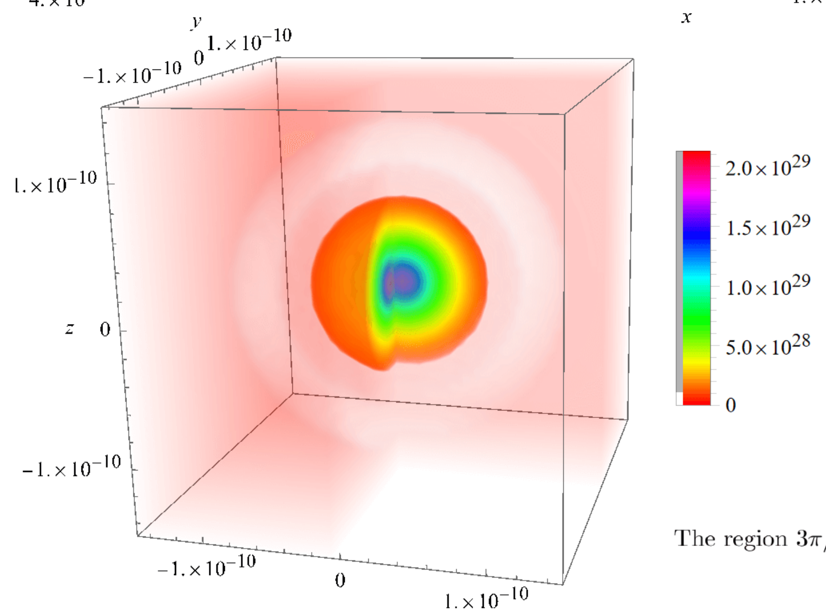


The region $3\pi/2 < \phi < 2\pi$ is excluded.

$$|\Psi_{200}(r,\theta,\phi,t)|^2$$

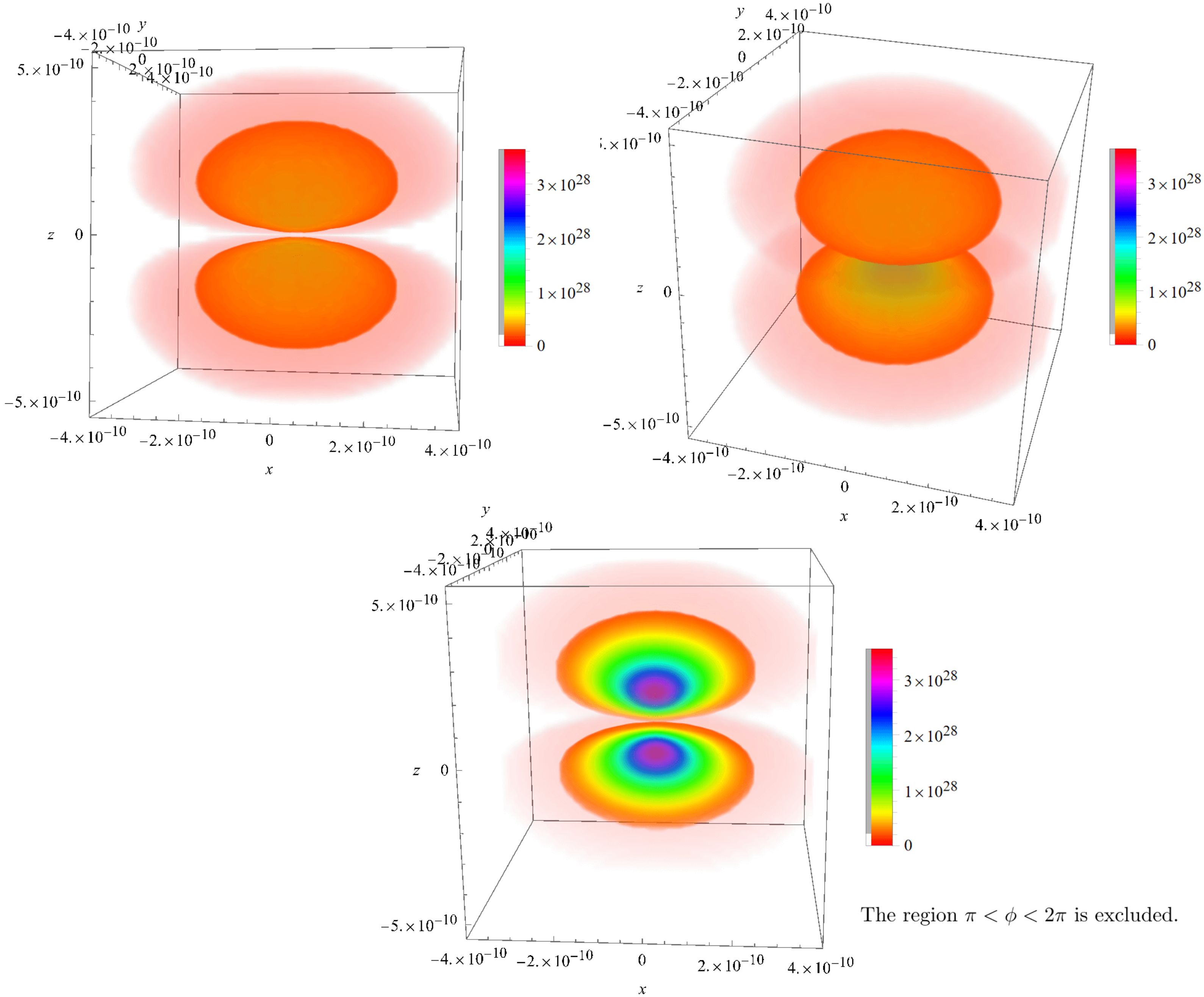






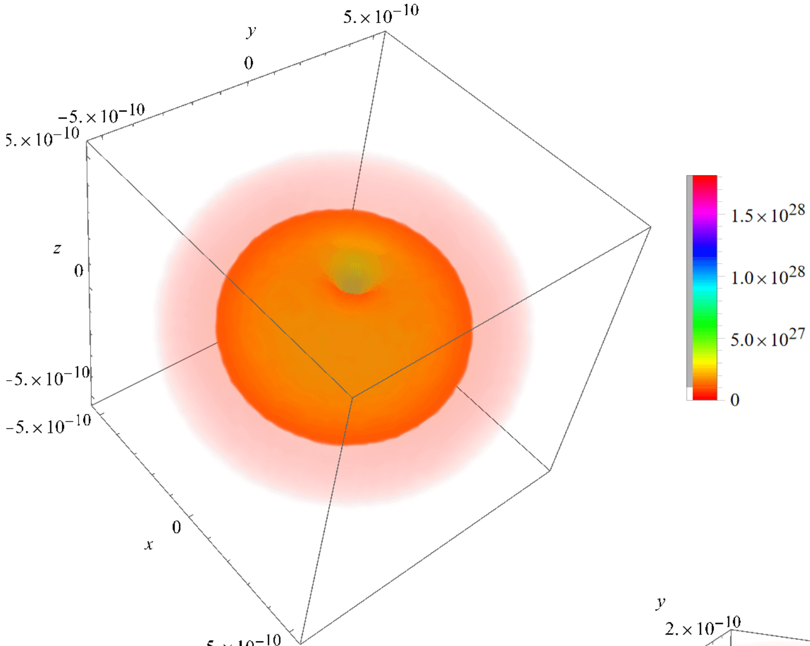
The region $3\pi/2 < \phi < 2\pi$ is excluded.

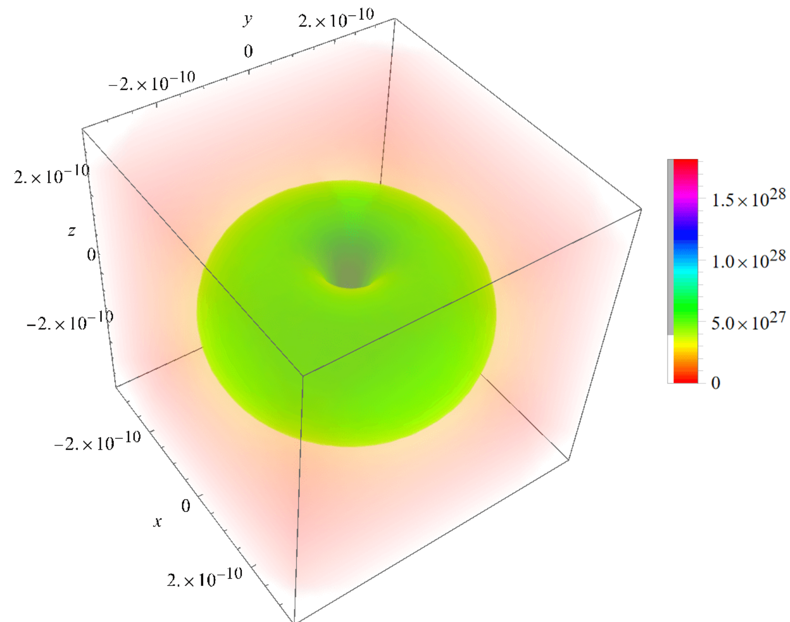
$|\Psi_{210}(r,\theta,\phi,t)|^2$

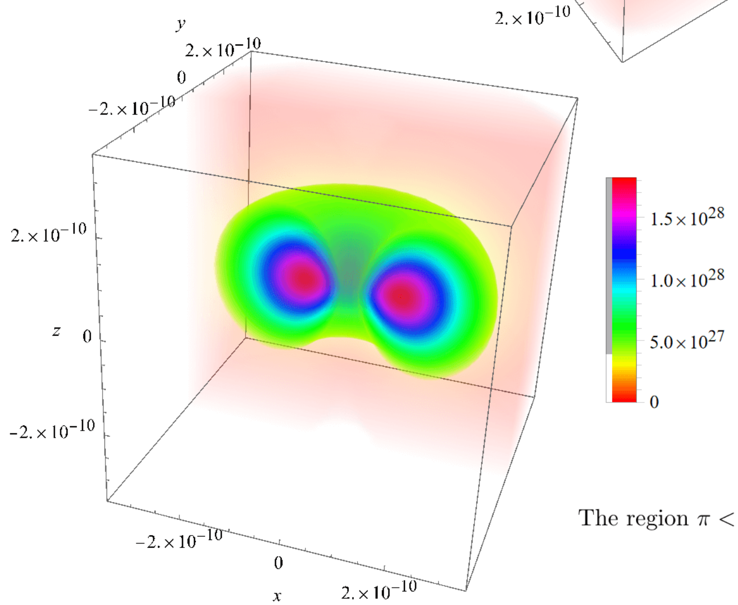


The region $\pi < \phi < 2\pi$ is excluded.

$$\left|\Psi_{21\pm1}(r,\theta,\phi,t)\right|^2$$

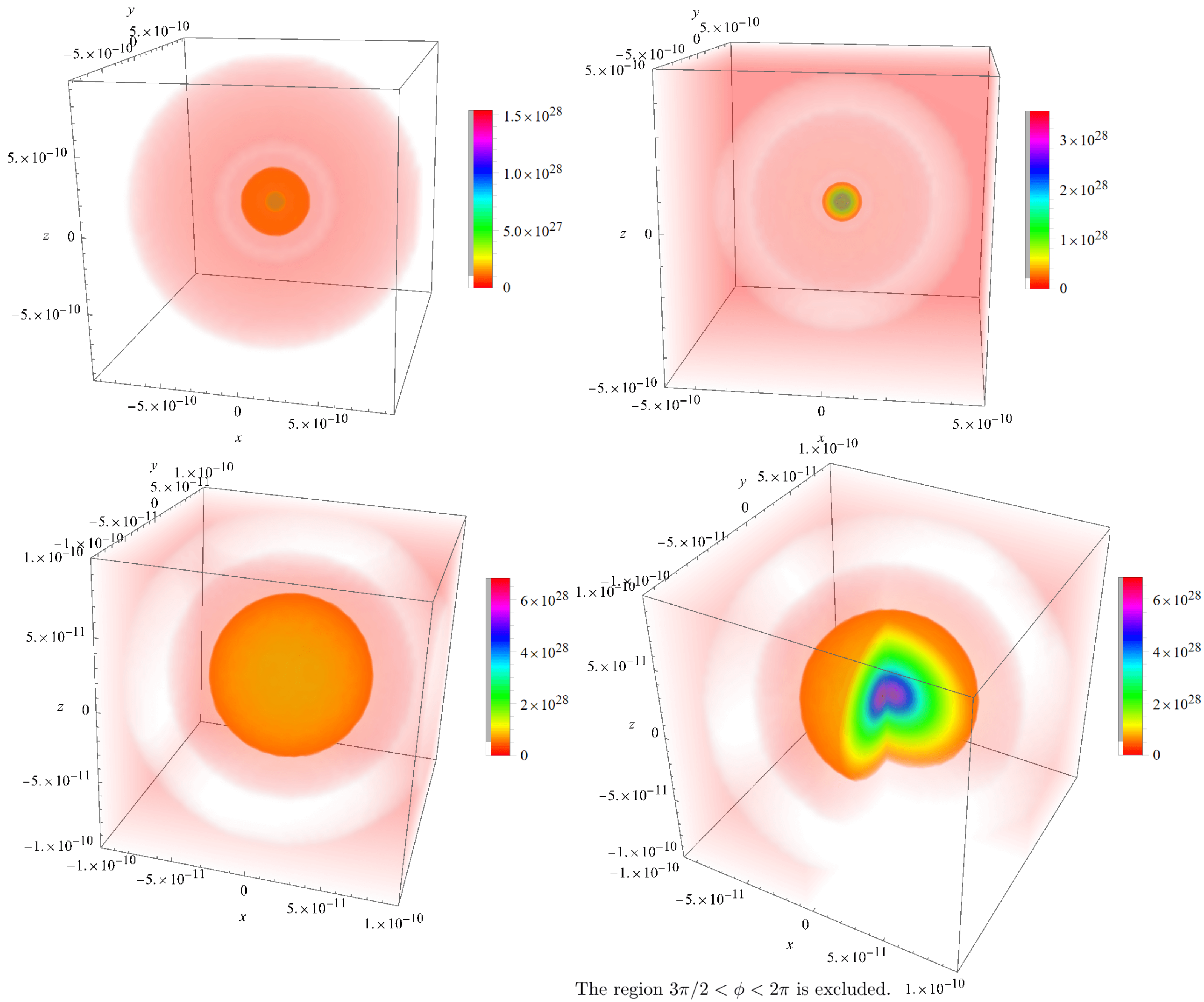






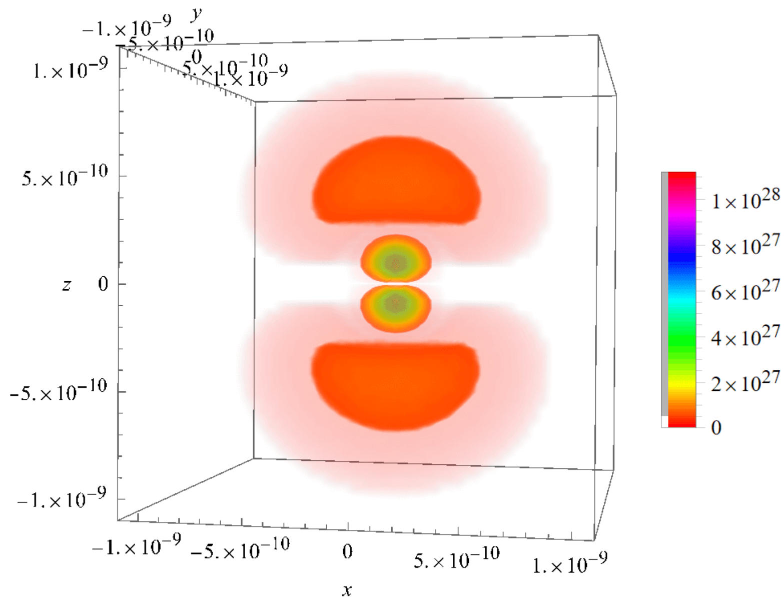
The region $\pi < \phi < 2\pi$ is excluded.

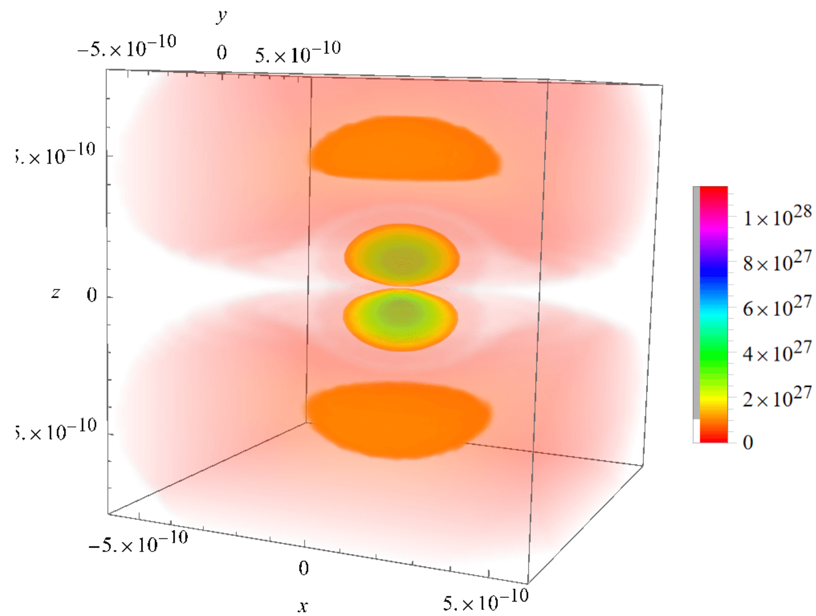
$$|\Psi_{300}(r, \theta, \phi, t)|^2$$

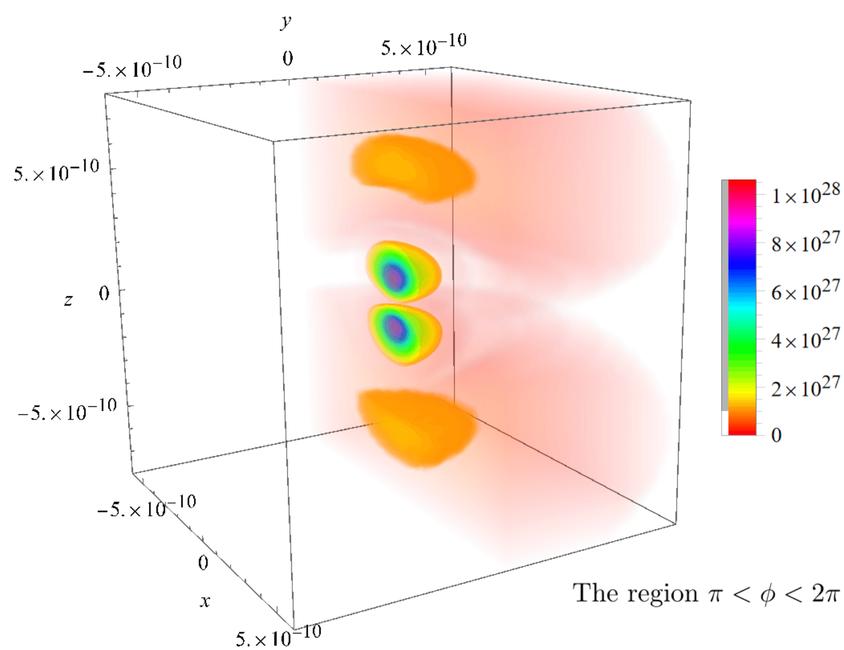


The region $3\pi/2 < \phi < 2\pi$ is excluded.

$$|\Psi_{310}(r, \theta, \phi, t)|^2$$

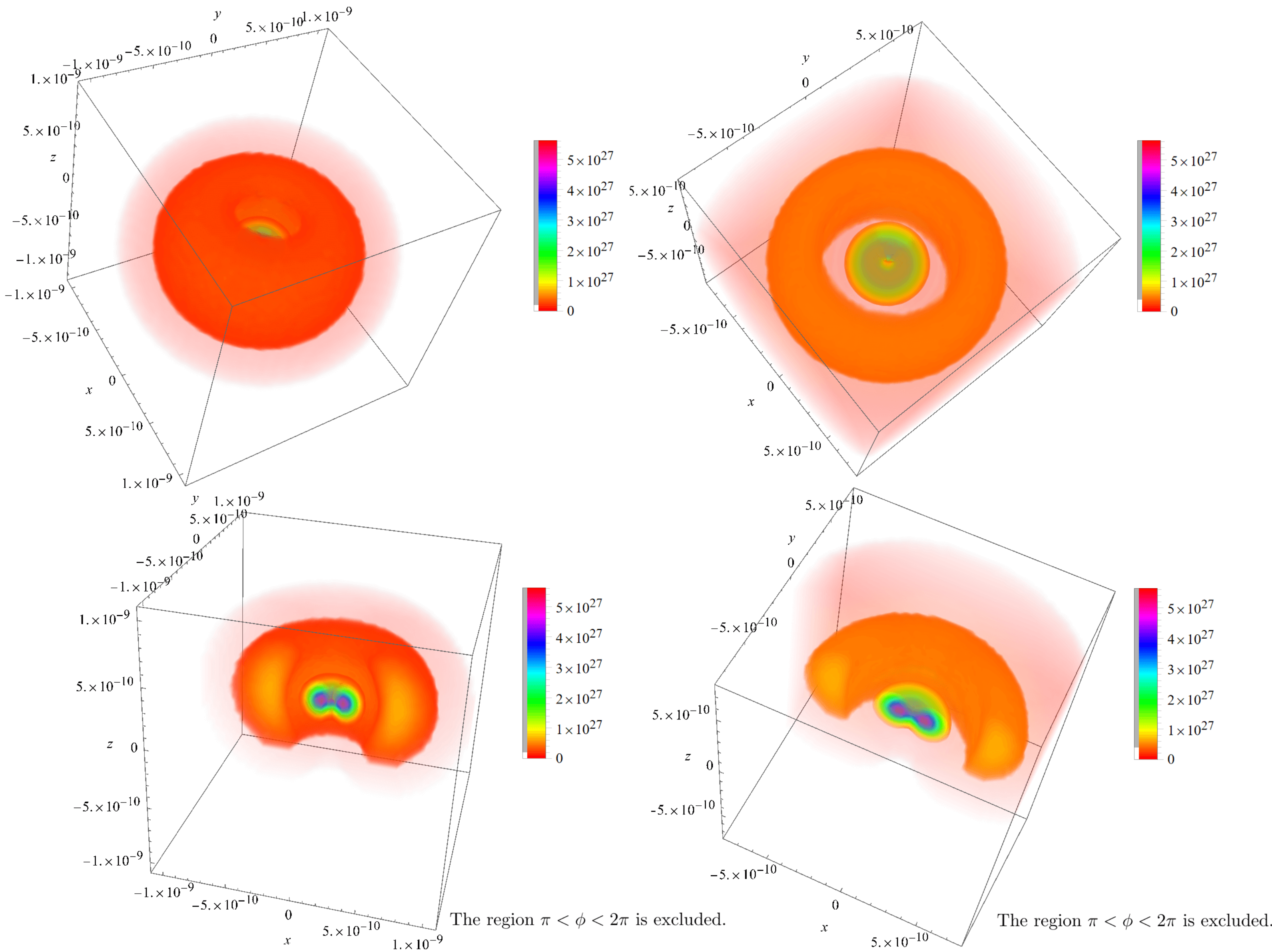






The region $\pi < \phi < 2\pi$ is excluded.

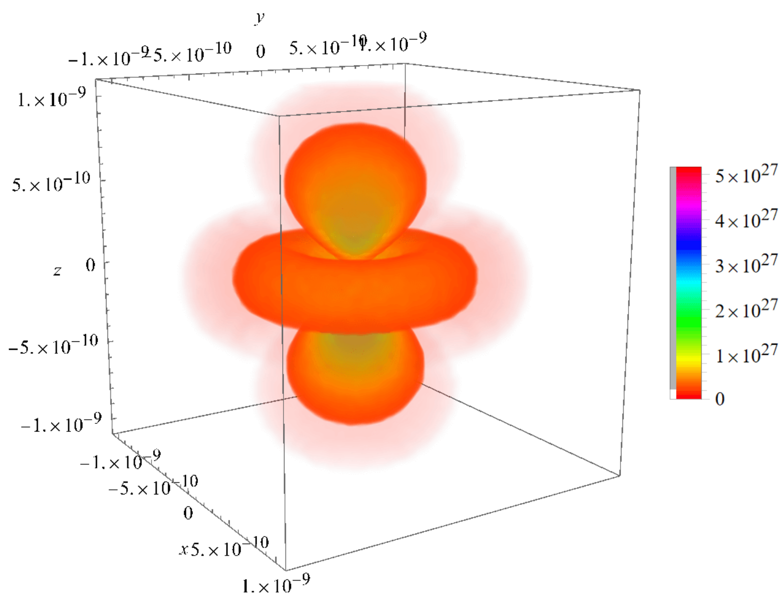
$$|\Psi_{31\pm1}(r,\theta,\phi,t)|^2$$

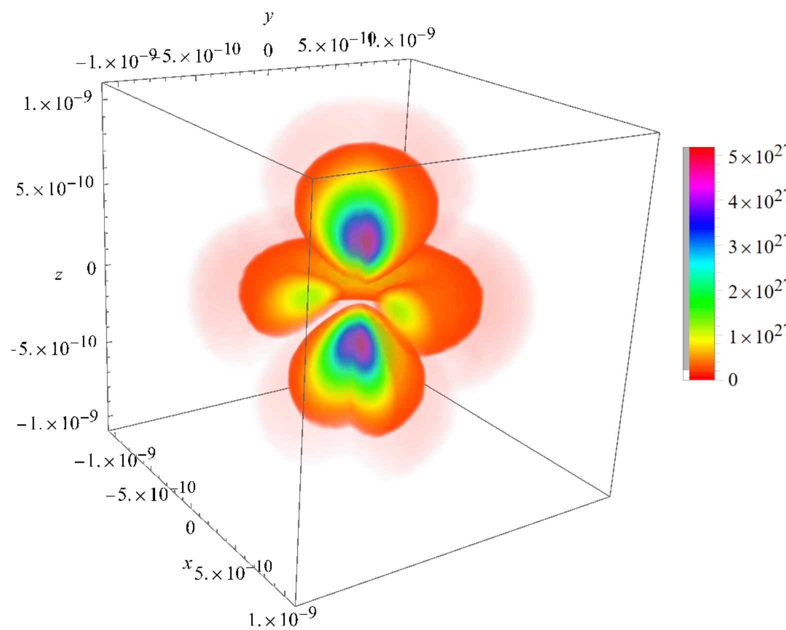


The region $\pi < \phi < 2\pi$ is excluded.

The region $\pi < \phi < 2\pi$ is excluded.

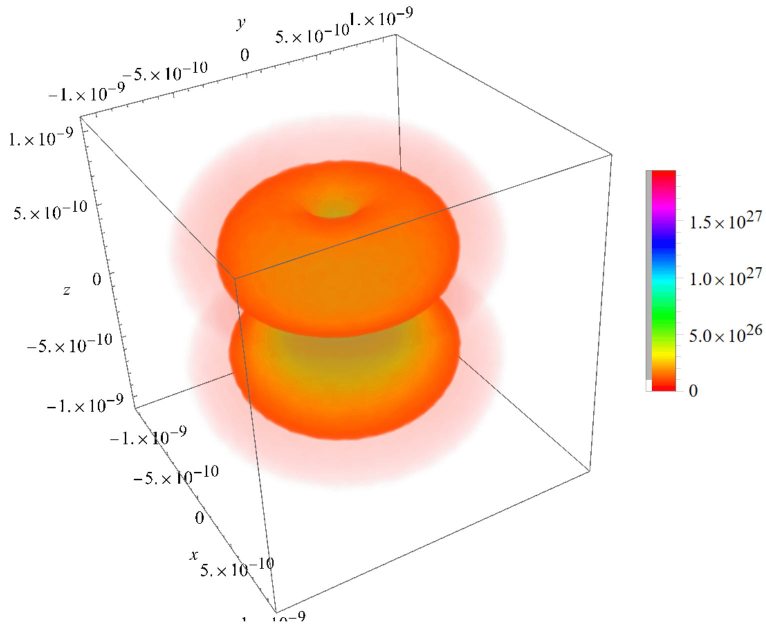
$$|\Psi_{320}(r,\theta,\phi,t)|^2$$

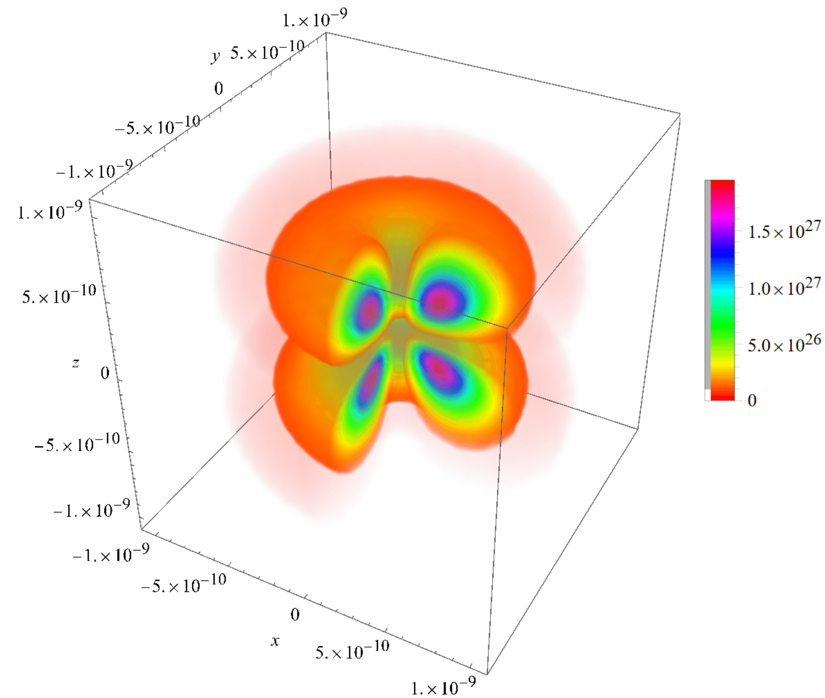




The region $3\pi/2 < \phi < 2\pi$ is excluded.

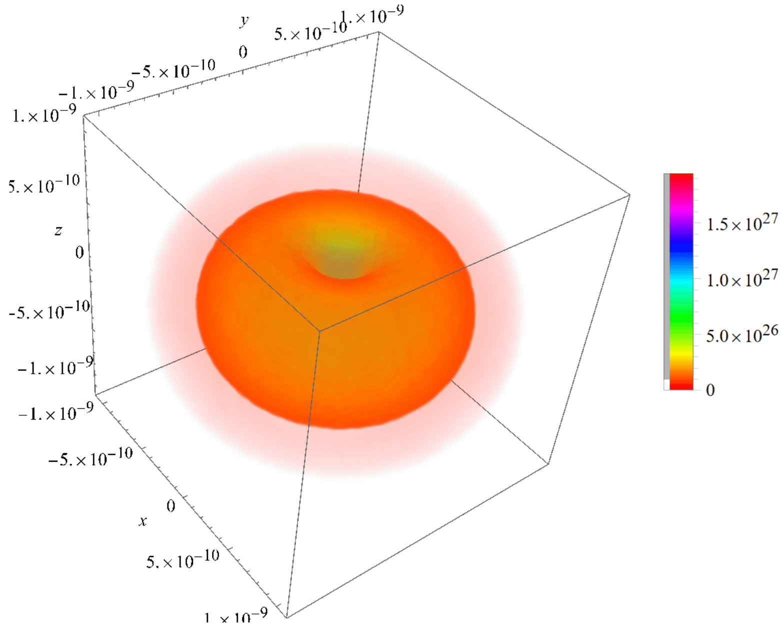
$|\Psi_{32\pm1}(r, \theta, \phi, t)|^2$

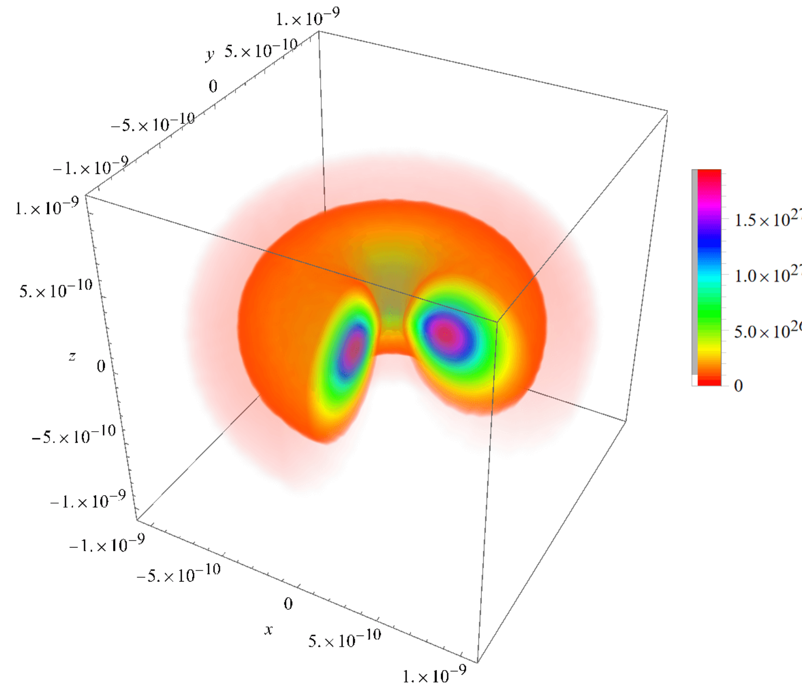




The region $3\pi/2 < \phi < 2\pi$ is excluded.

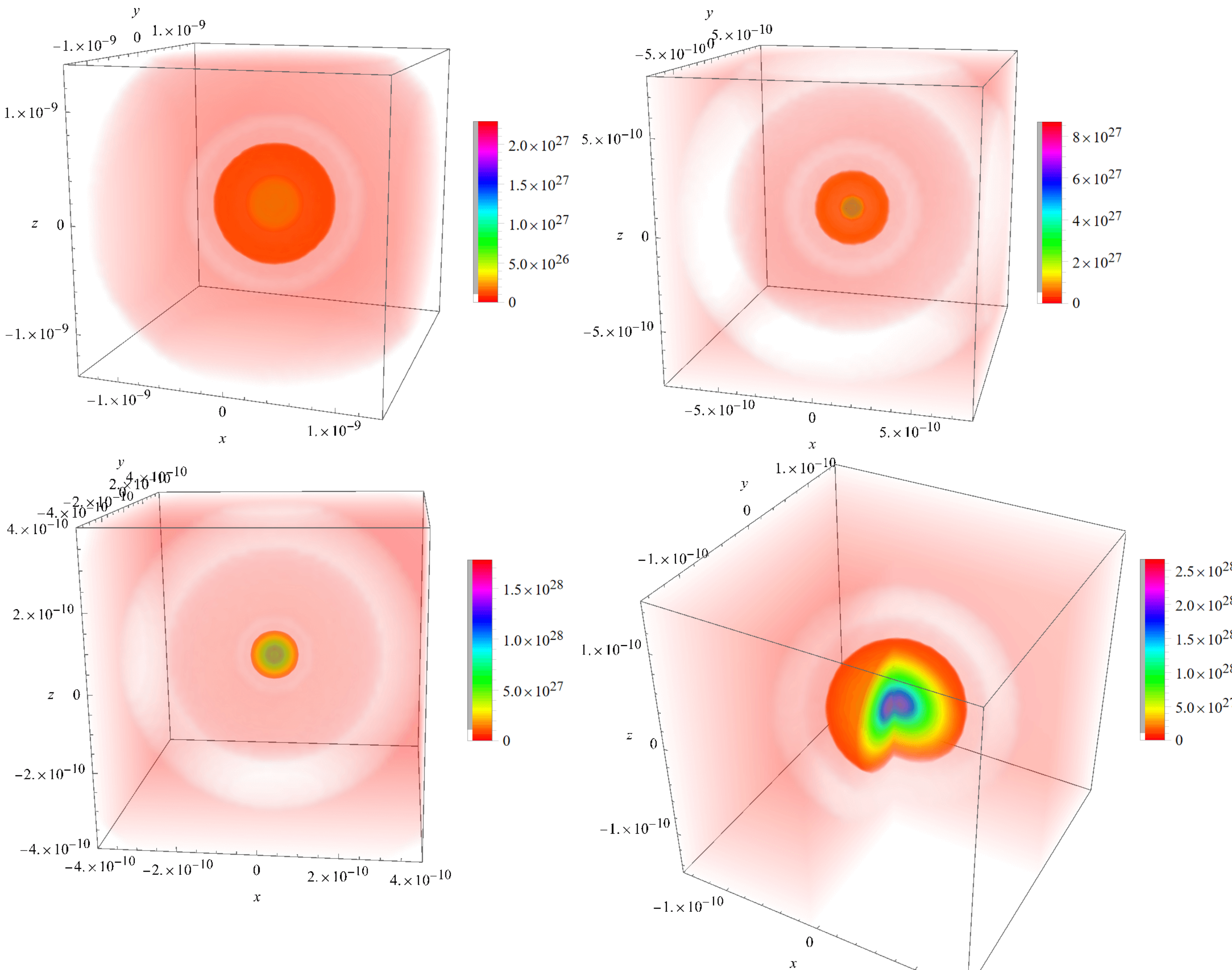
$$|\Psi_{32\pm2}(r,\theta,\phi,t)|^2$$





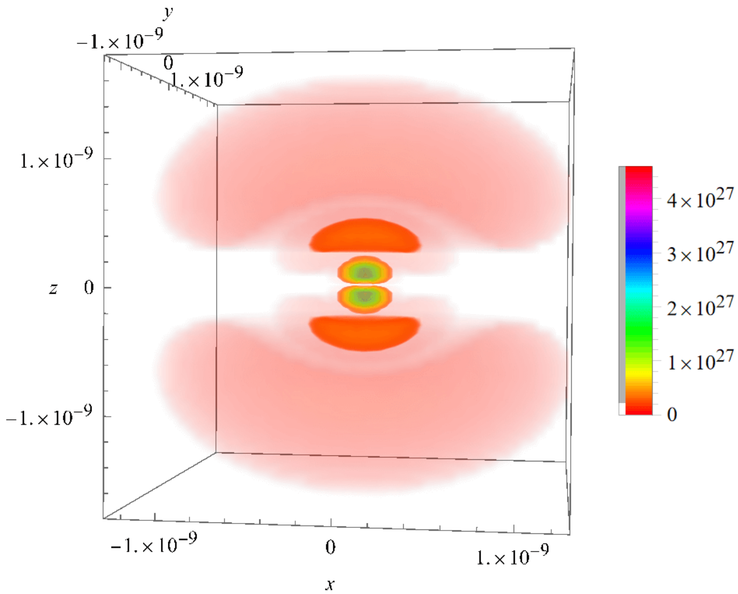
The region $3\pi/2 < \phi < 2\pi$ is excluded.

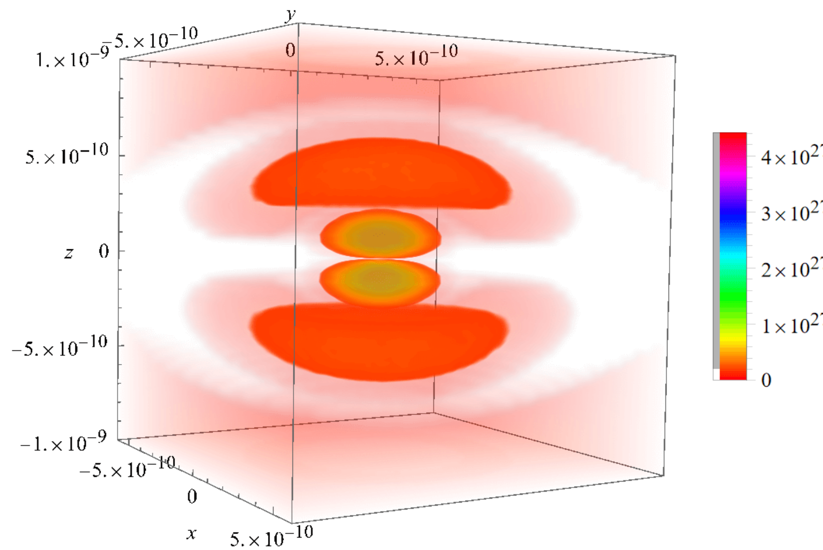
$$|\Psi_{400}(r, \theta, \phi, t)|^2$$

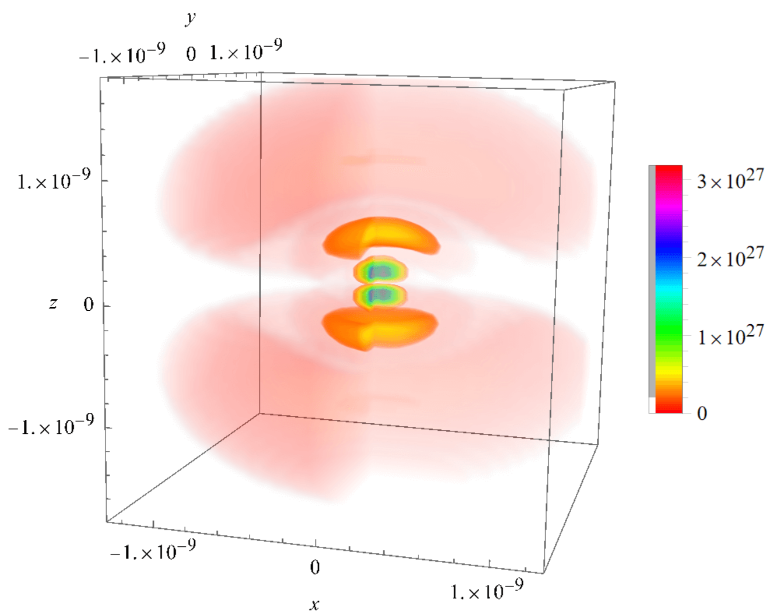


The region $3\pi/2 < \phi < 2\pi$ is excluded.

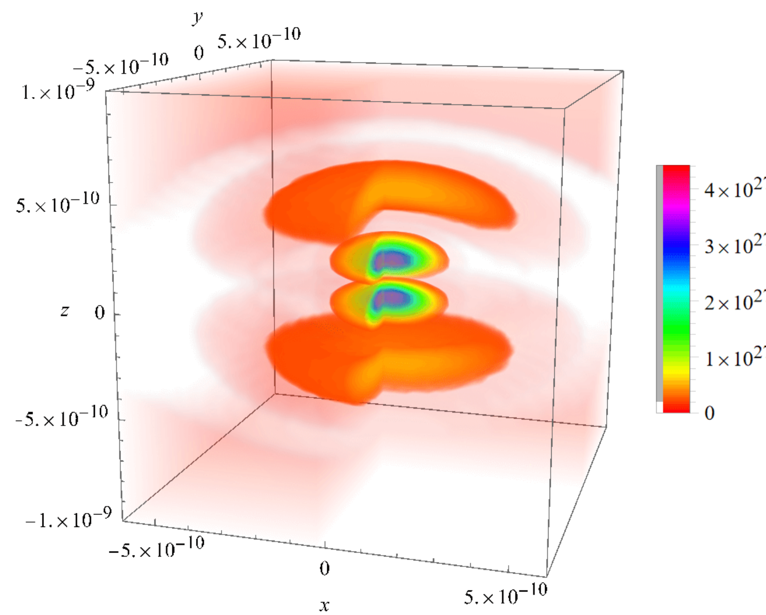
$$|\Psi_{410}(r, \theta, \phi, t)|^2$$





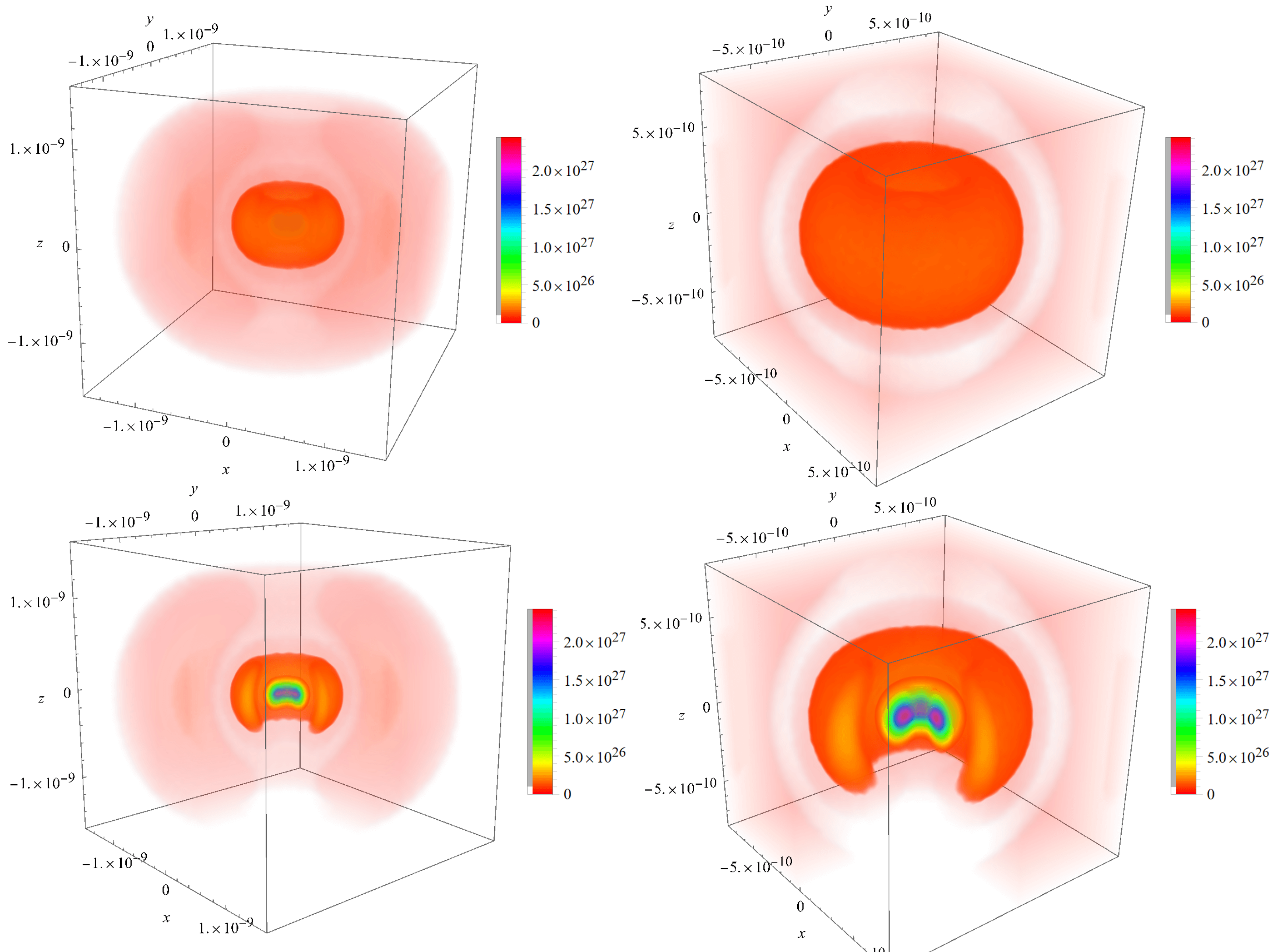


The region $3\pi/2 < \phi < 2\pi$ is excluded.



The region $3\pi/2 < \phi < 2\pi$ is excluded.

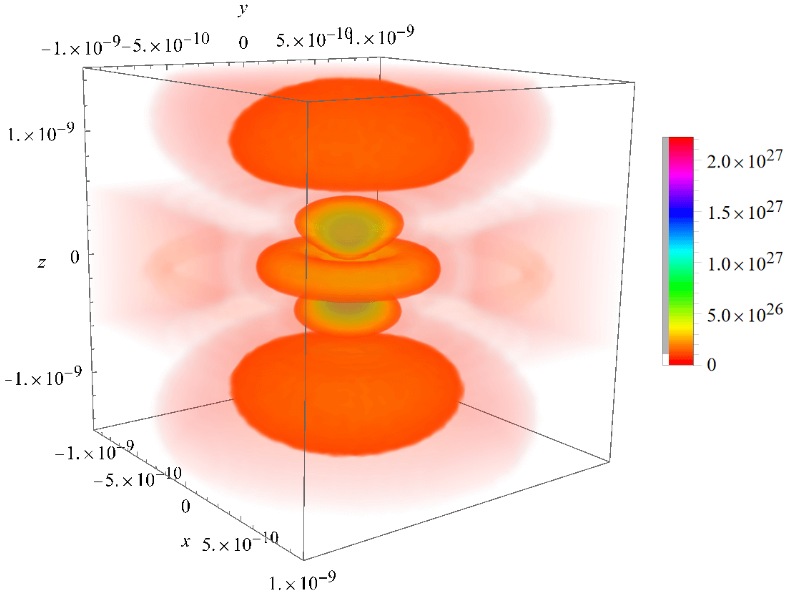
$$|\Psi_{41\pm1}(r,\theta,\phi,t)|^2$$

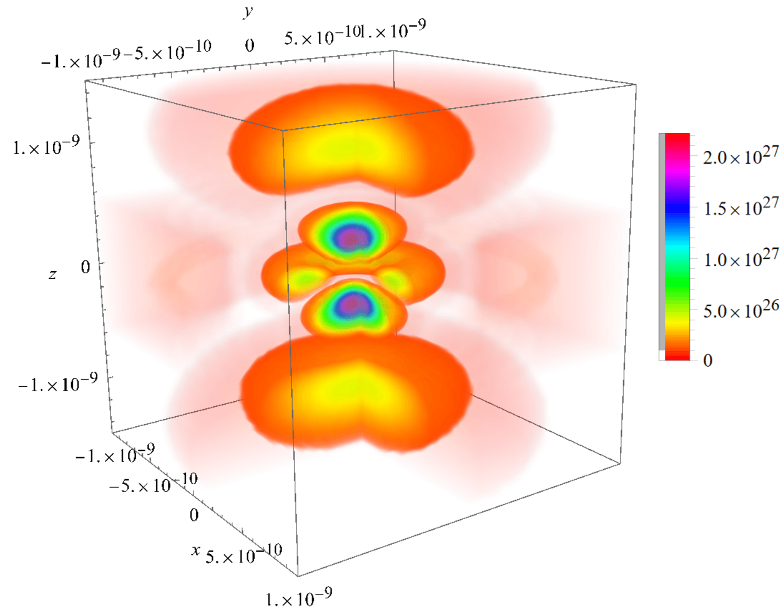


The region $3\pi/2 < \phi < 2\pi$ is excluded.

The region $3\pi/2 < \phi < 2\pi$ is excluded.

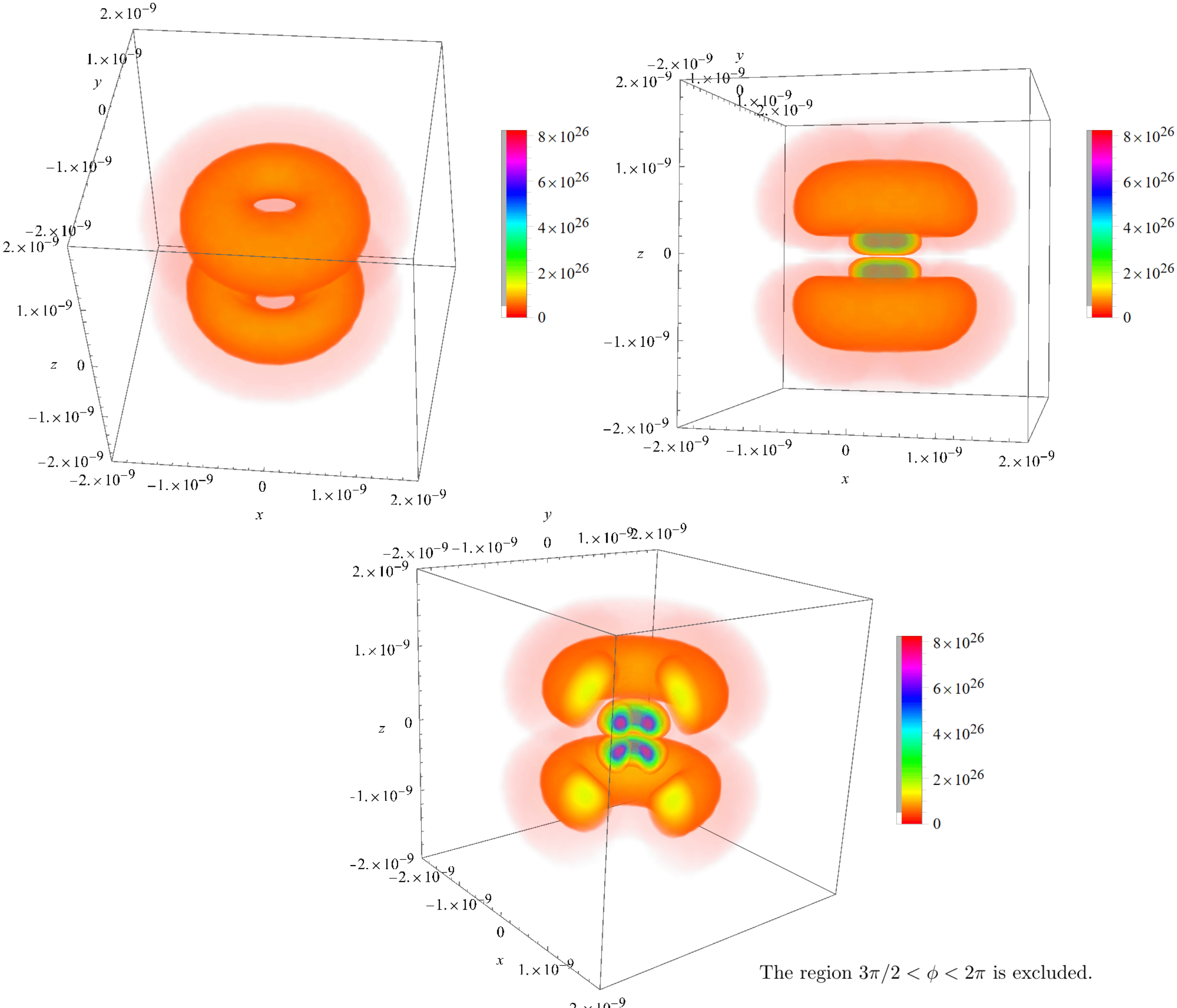
$$|\Psi_{420}(r, \theta, \phi, t)|^2$$





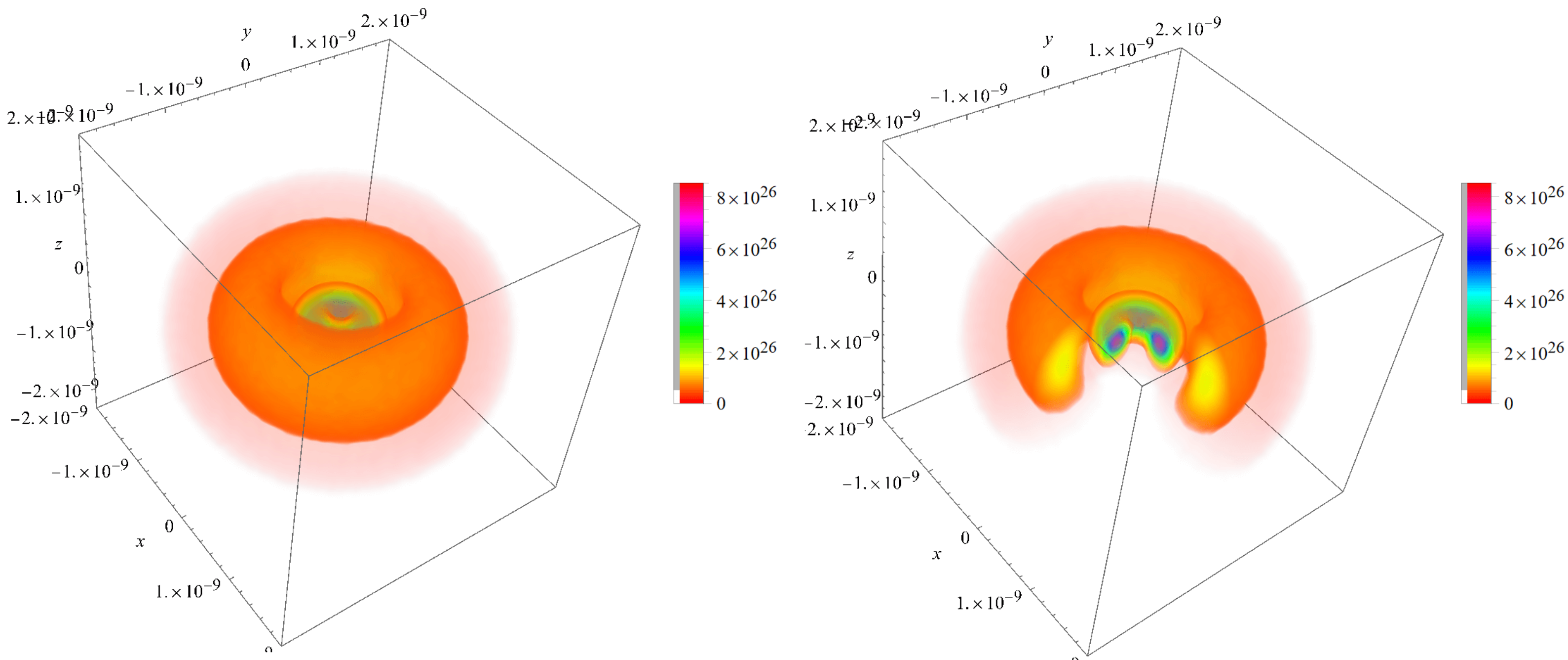
The region $3\pi/2 < \phi < 2\pi$ is excluded.

$$|\Psi_{42\pm1}(r,\theta,\phi,t)|^2$$



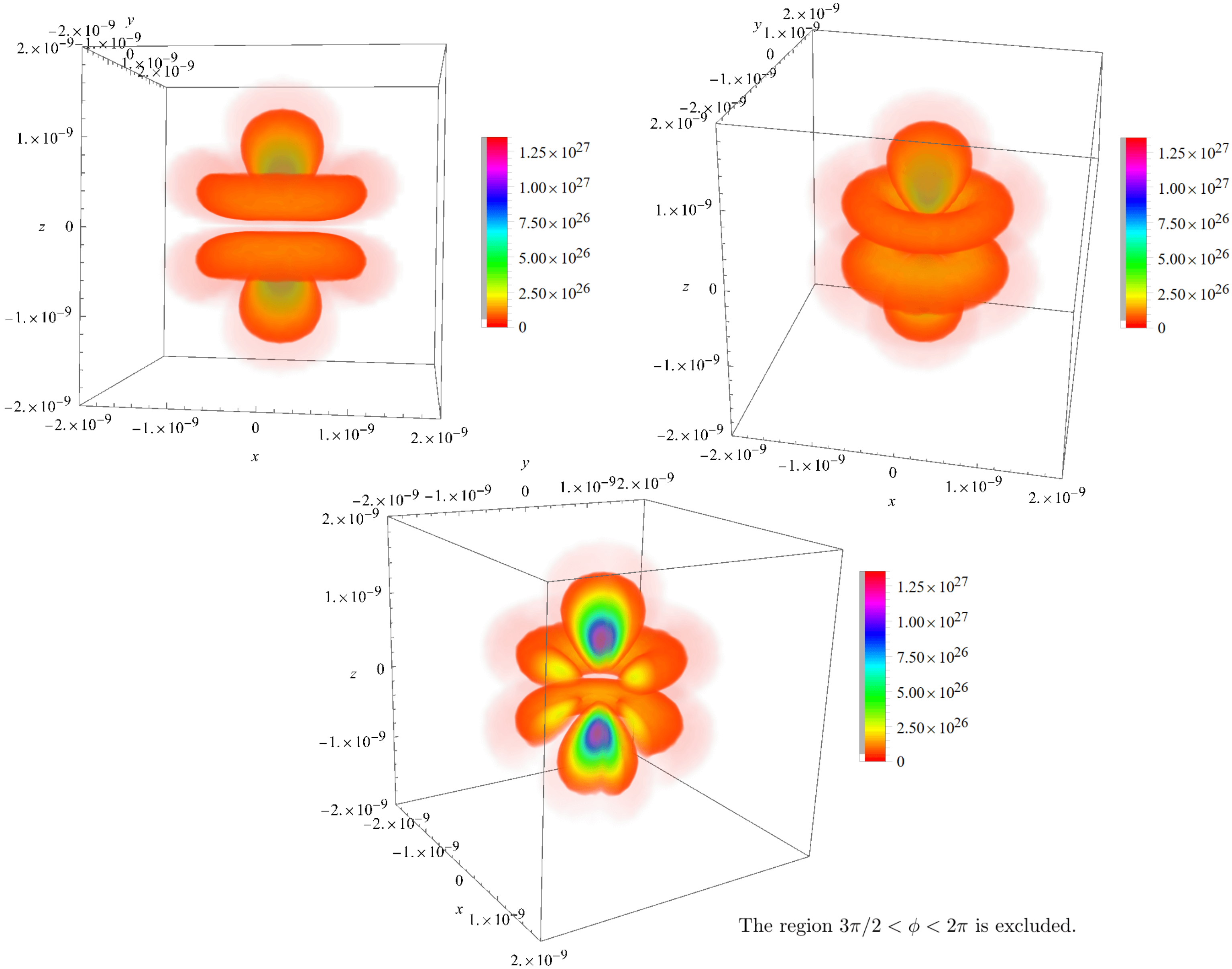
The region $3\pi/2 < \phi < 2\pi$ is excluded.

$|\Psi_{42\pm2}(r,\theta,\phi,t)|^2$



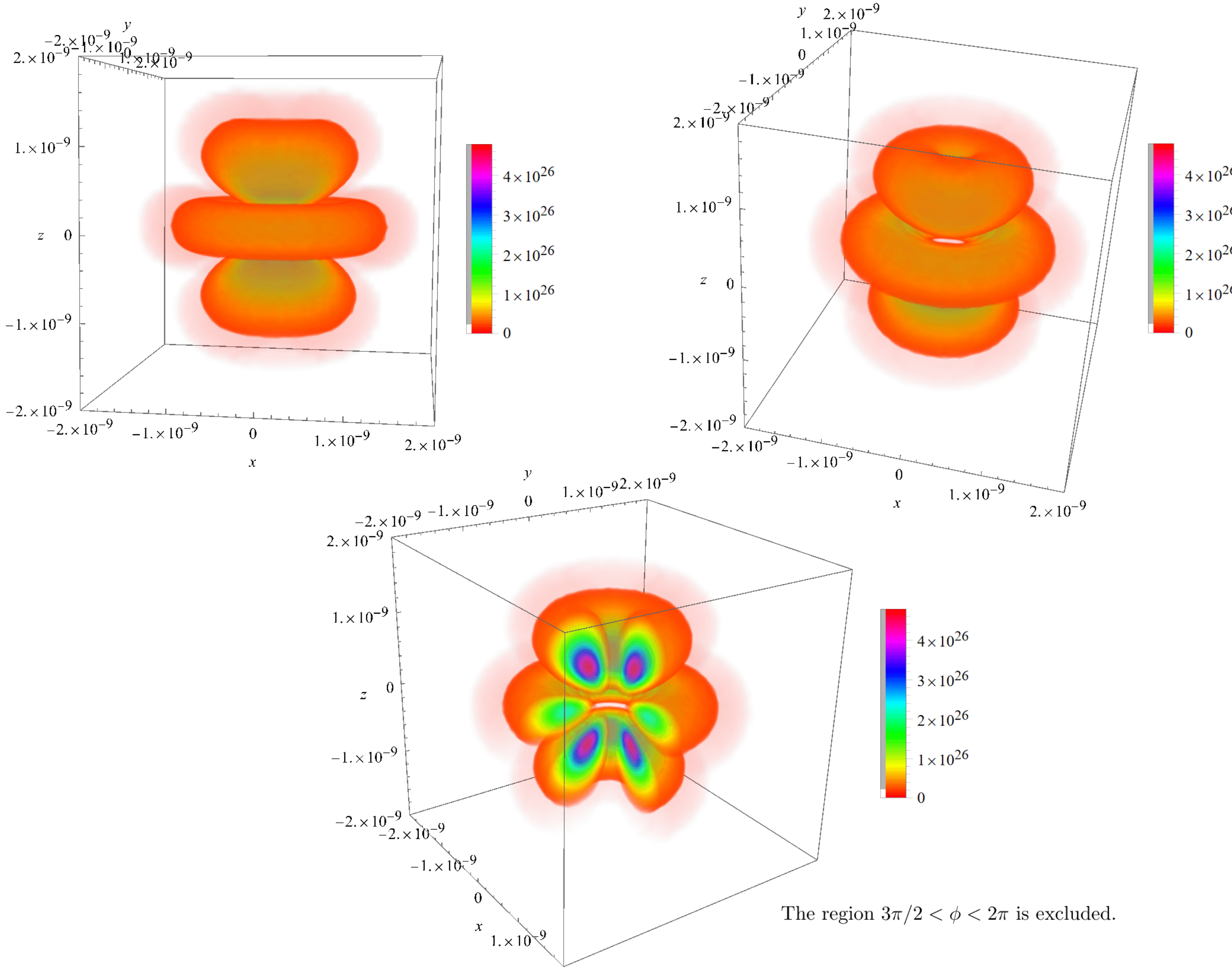
The region $3\pi/2 < \phi < 2\pi$ is excluded.

$$|\Psi_{430}(r, \theta, \phi, t)|^2$$



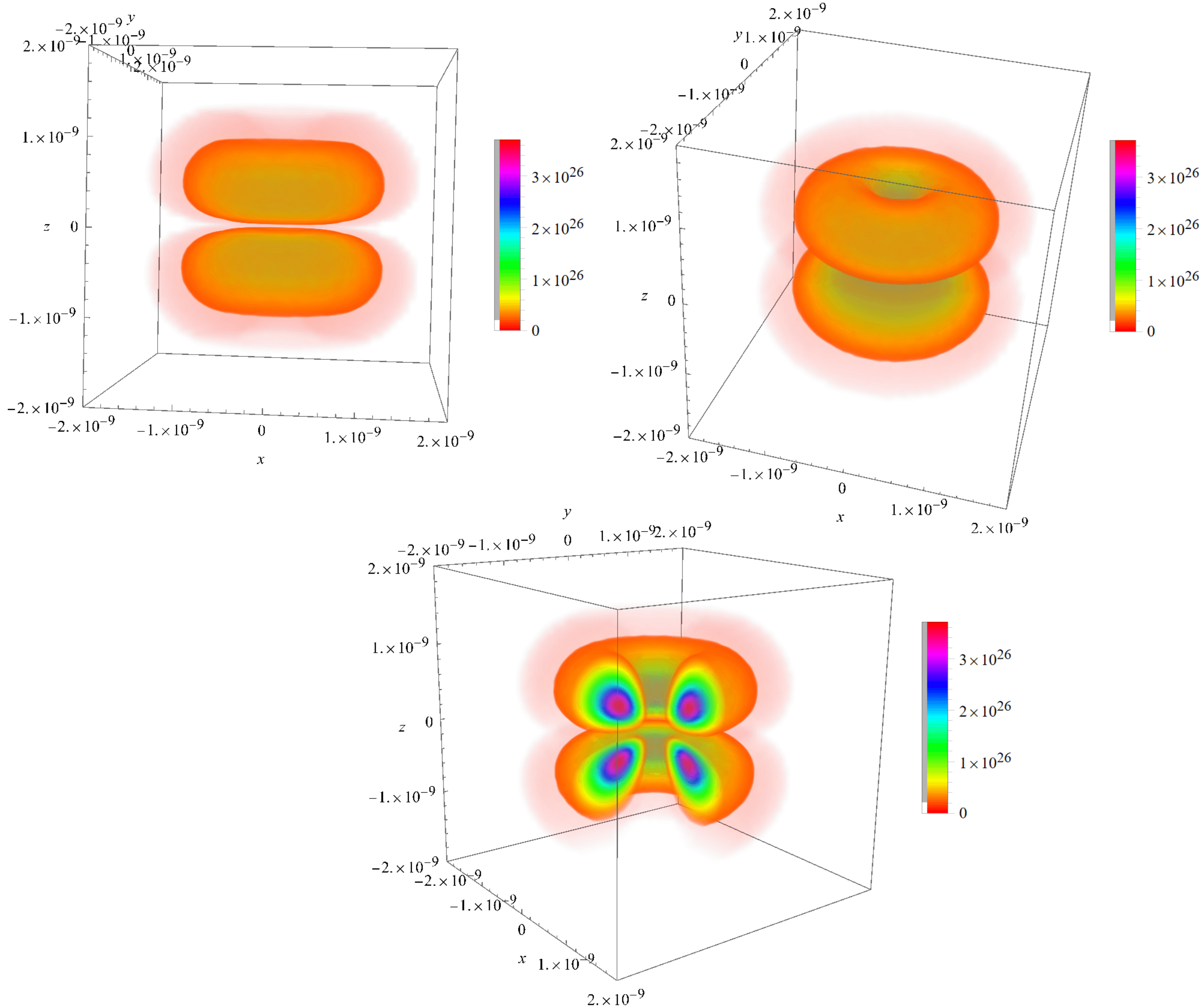
The region $3\pi/2 < \phi < 2\pi$ is excluded.

$$|\Psi_{43\pm1}(r, \theta, \phi, t)|^2$$

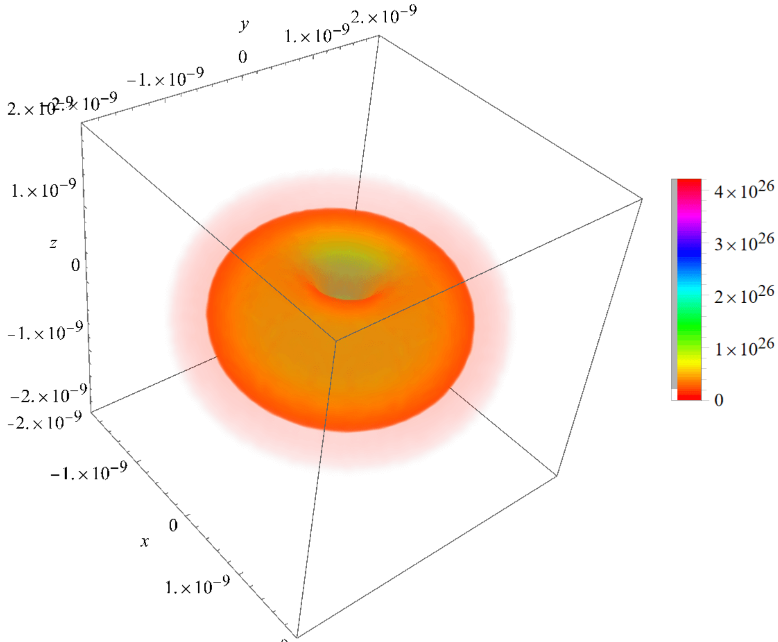


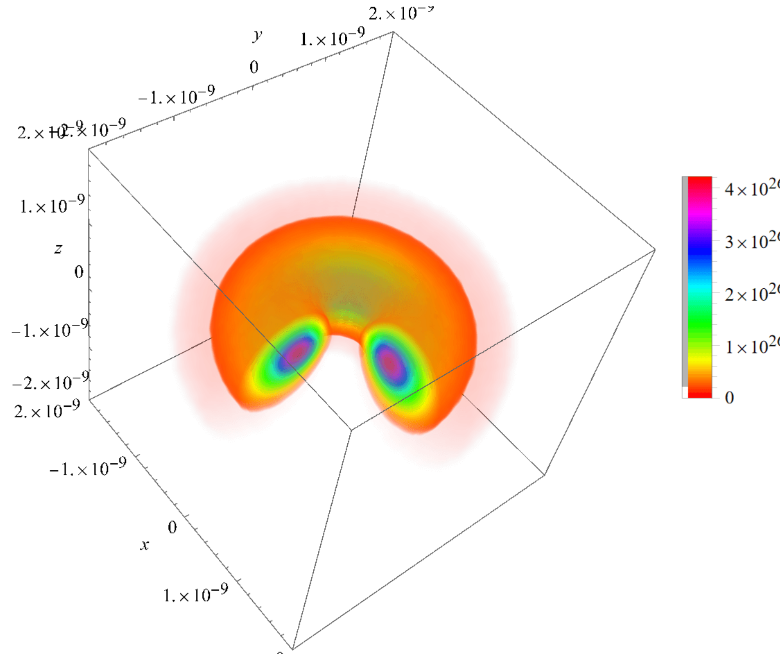
The region $3\pi/2 < \phi < 2\pi$ is excluded.

$$|\Psi_{43\pm2}(r,\theta,\phi,t)|^2$$



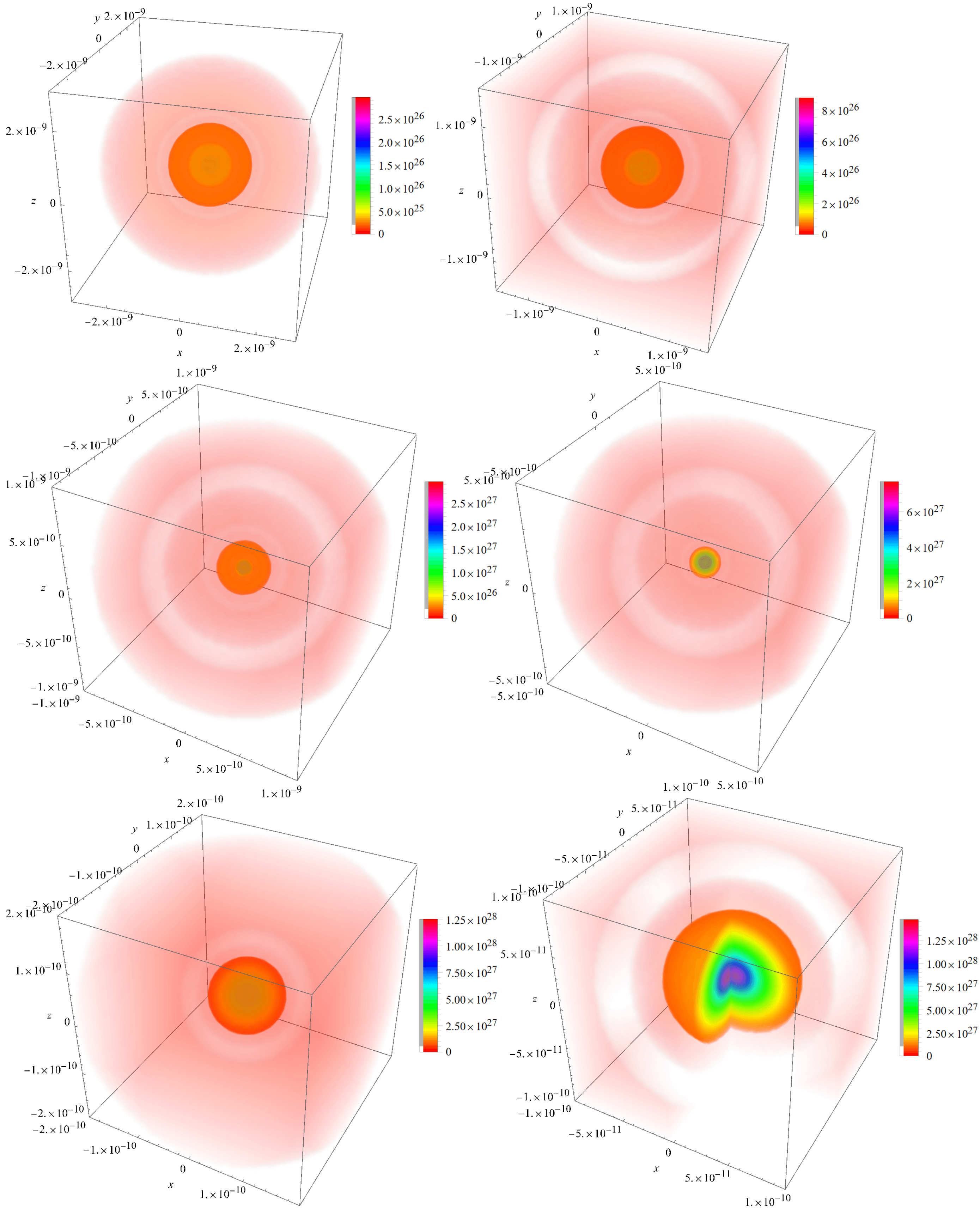
$$|\Psi_{43\pm3}(r,\theta,\phi,t)|^2$$





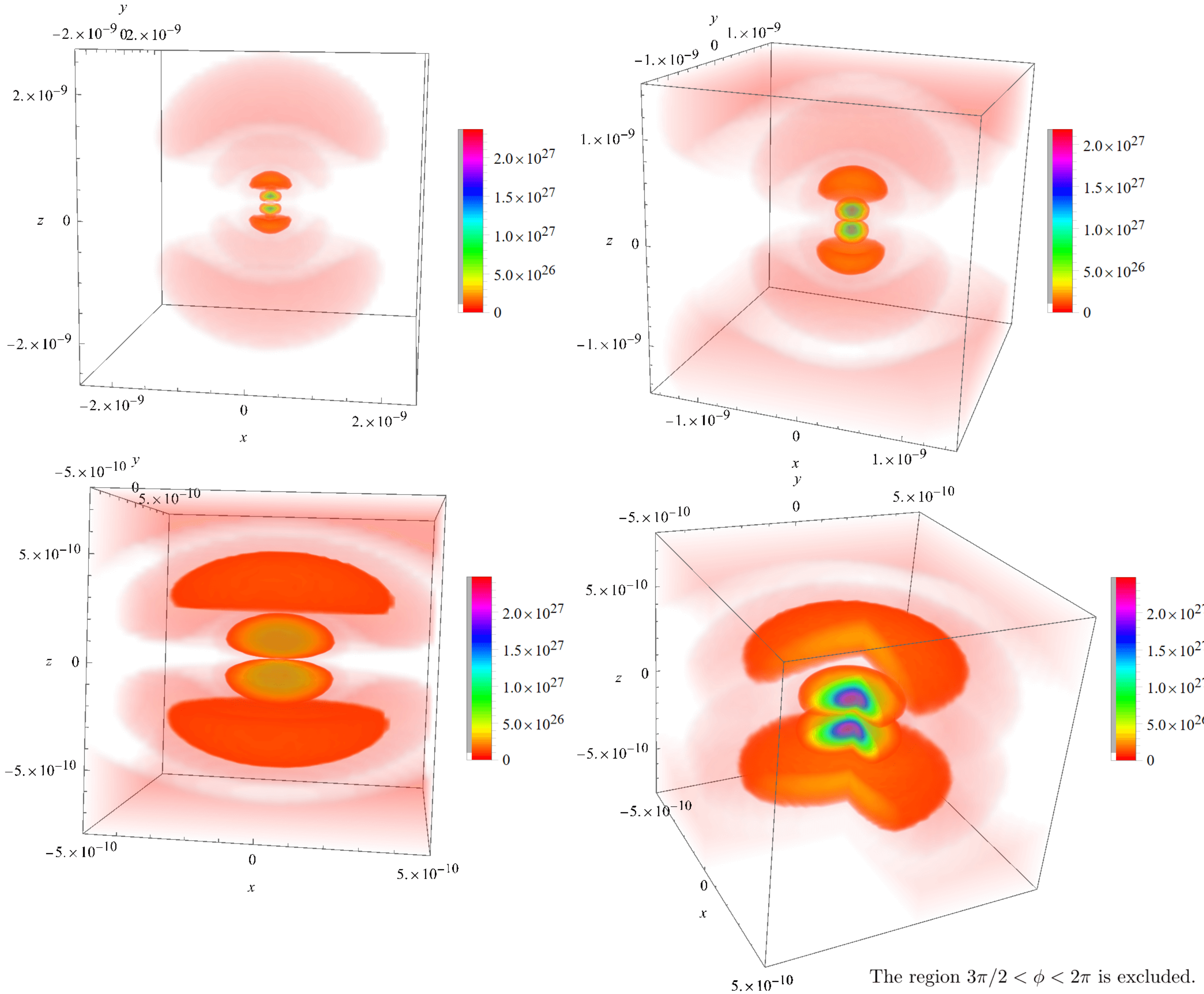
The region $3\pi/2 < \phi < 2\pi$ is excluded.

$|\Psi_{500}(r, \theta, \phi, t)|^2$



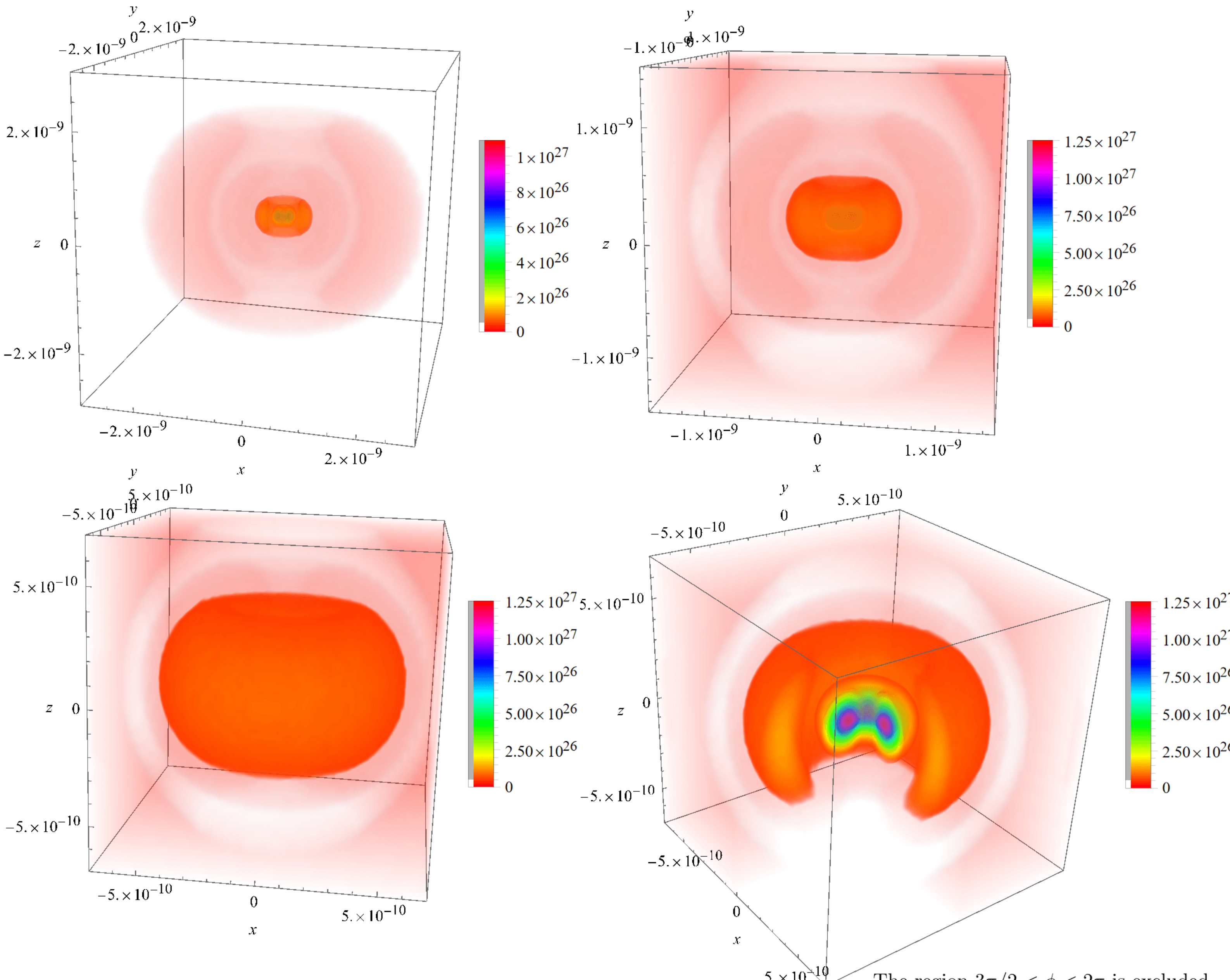
The region $3\pi/2 < \phi < 2\pi$ is excluded.

$$|\Psi_{510}(r, \theta, \phi, t)|^2$$



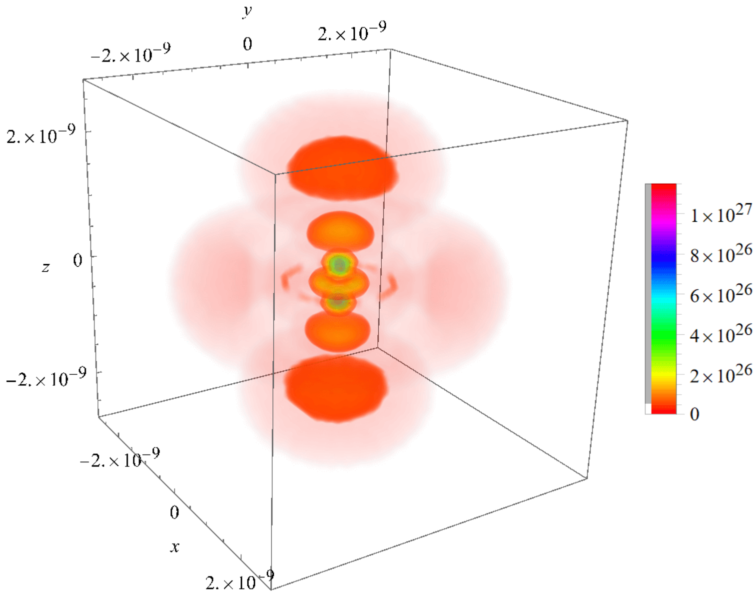
The region $3\pi/2 < \phi < 2\pi$ is excluded.

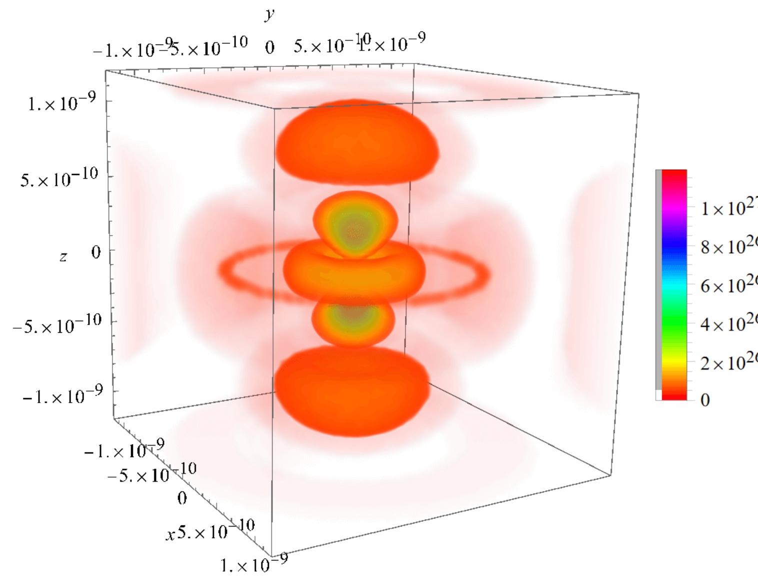
$$|\Psi_{51\pm1}(r,\theta,\phi,t)|^2$$

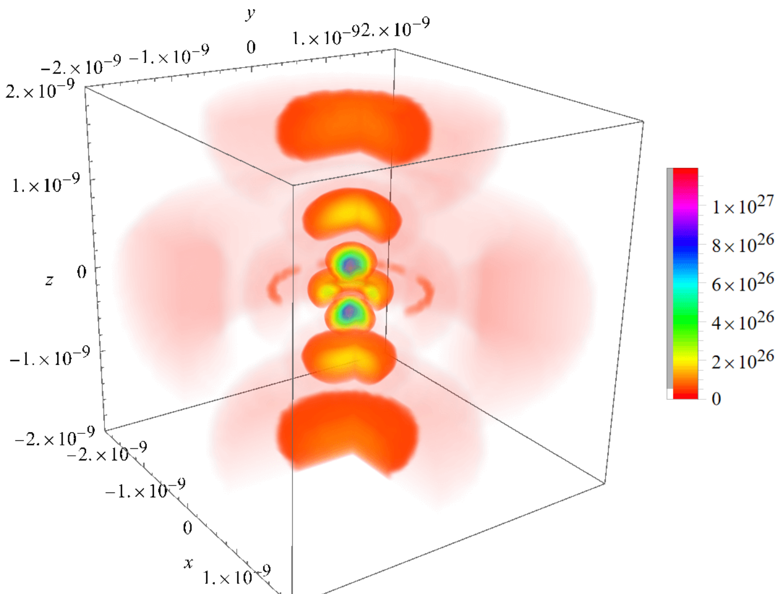


The region $3\pi/2 < \phi < 2\pi$ is excluded.

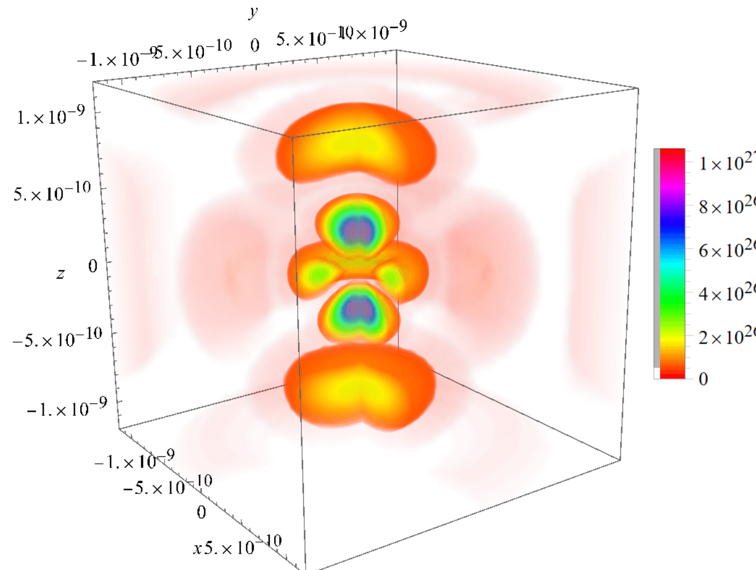
$$|\Psi_{520}(r,\theta,\phi,t)|^2$$





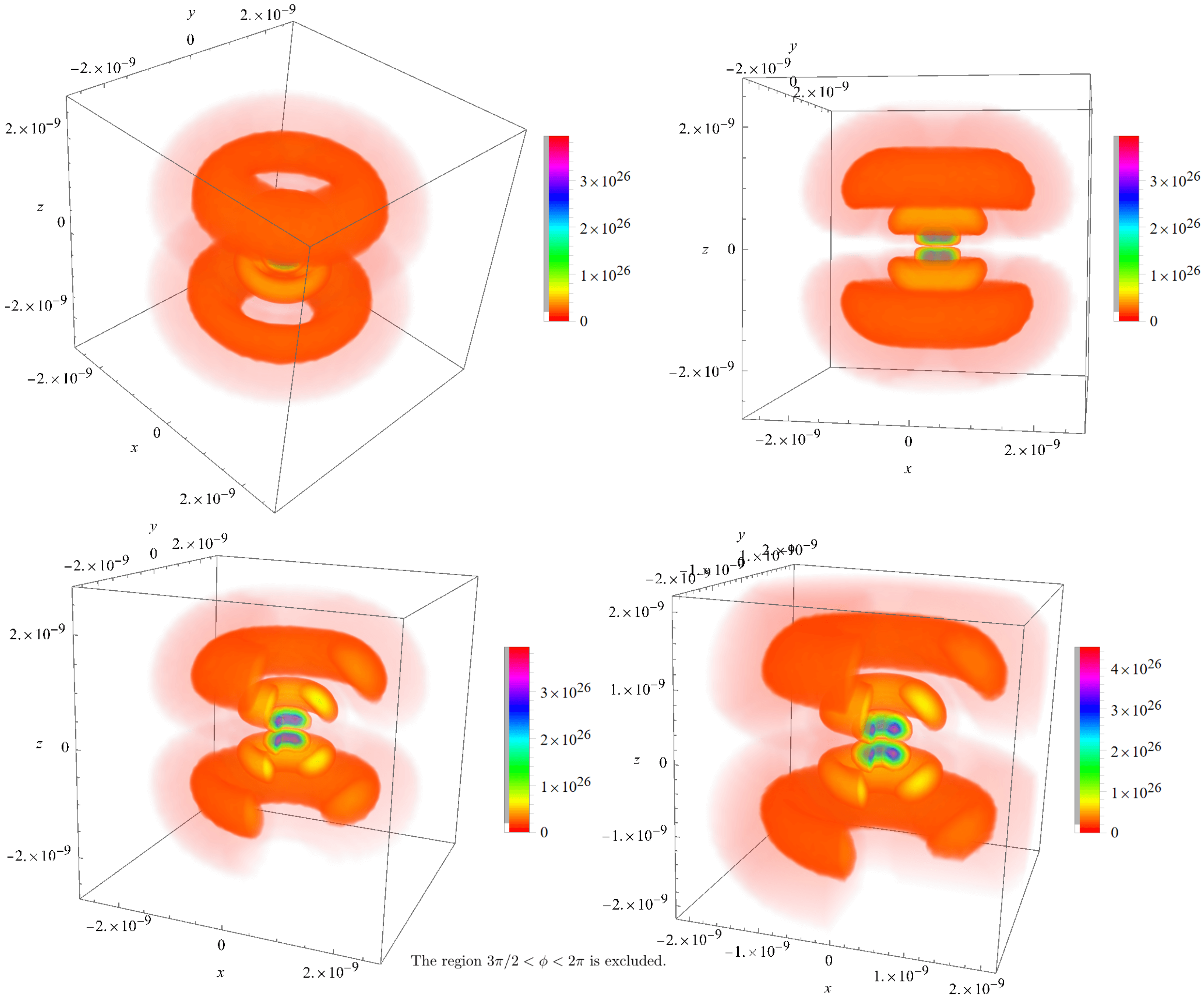


The region $3\pi/2 < \phi < 2\pi$ is excluded.



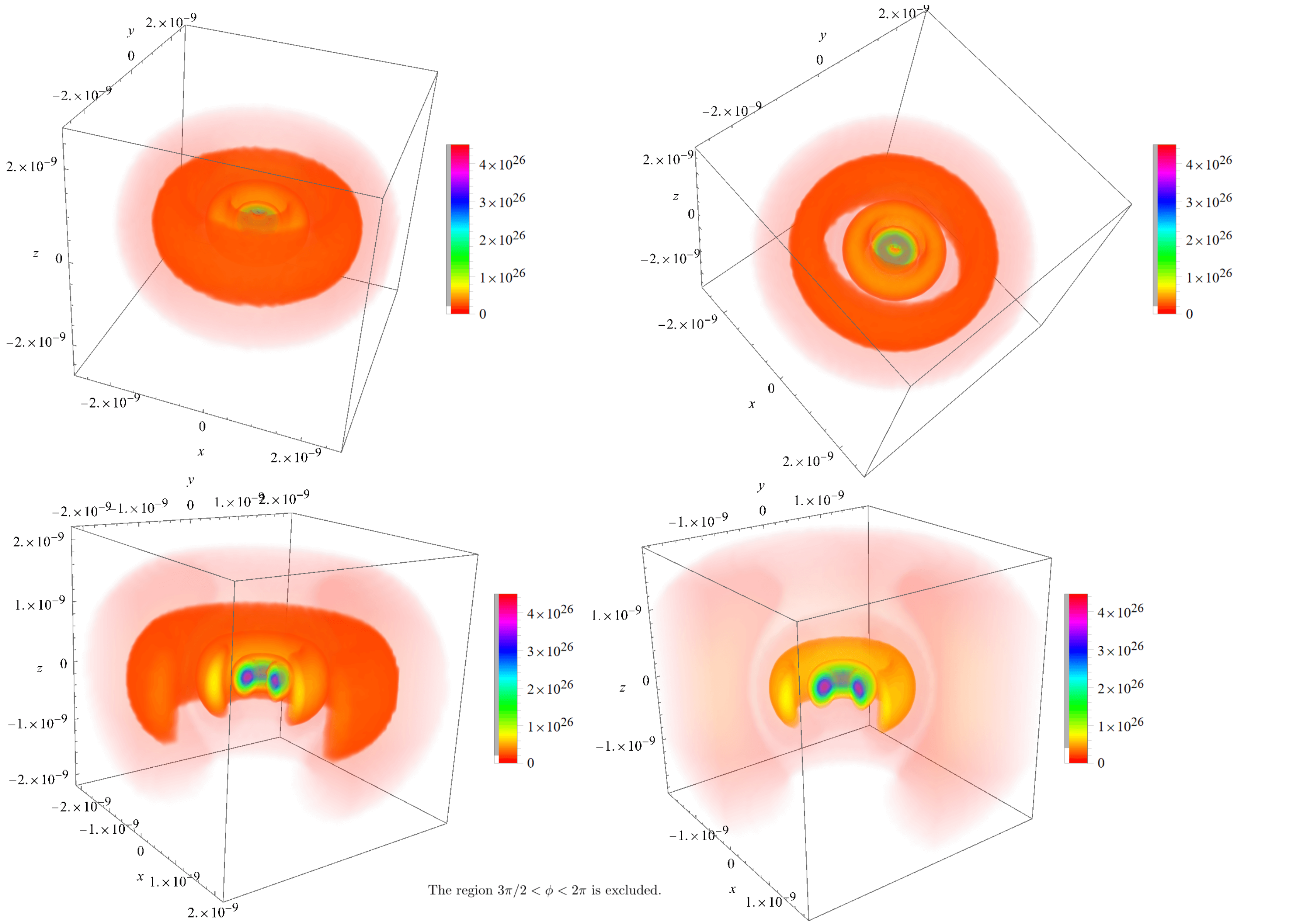
The region $3\pi/2 < \phi < 2\pi$ is excluded.

$$|\Psi_{52\pm1}(r,\theta,\phi,t)|^2$$



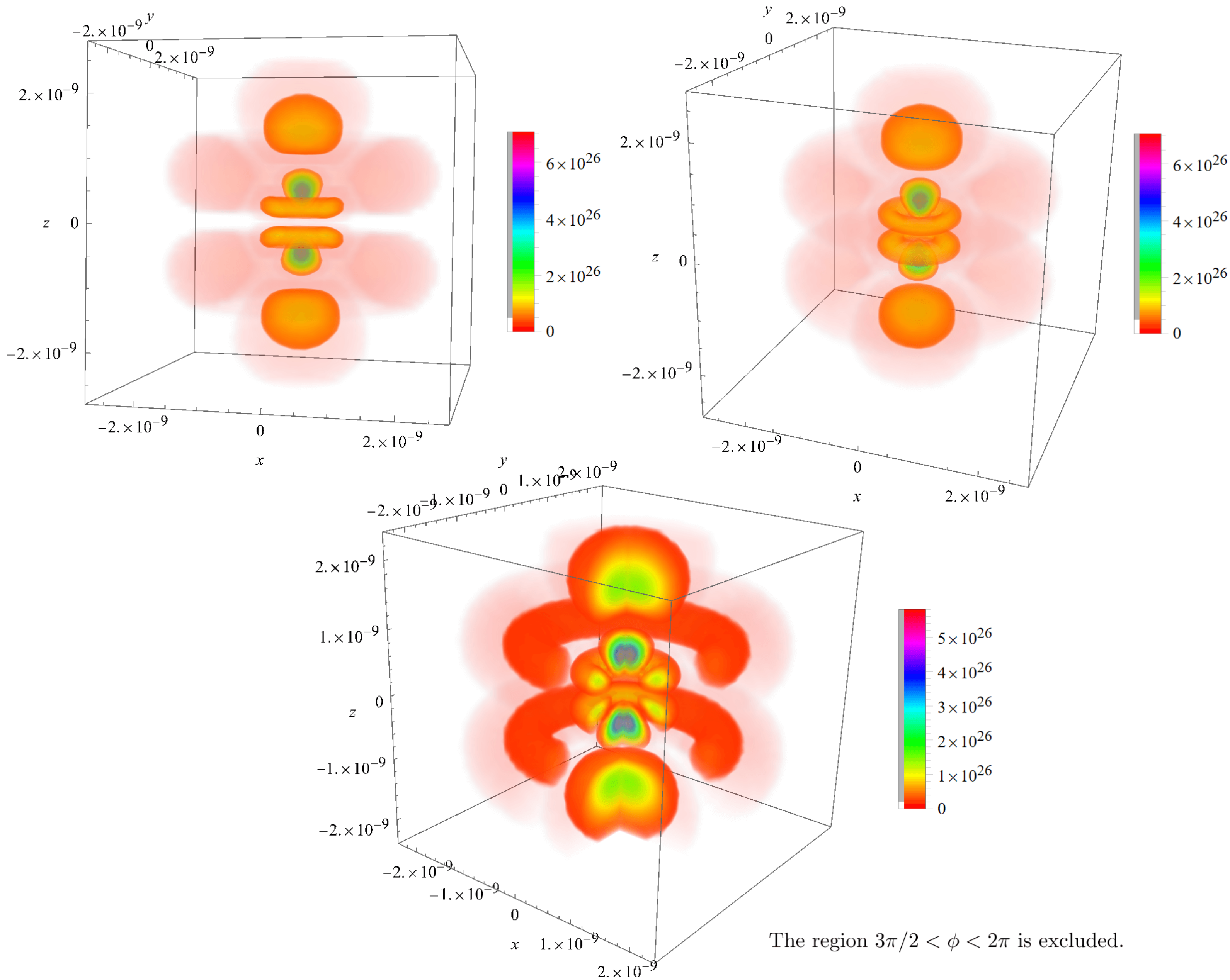
The region $3\pi/2 < \phi < 2\pi$ is excluded.

$$|\Psi_{52\pm2}(r,\theta,\phi,t)|^2$$



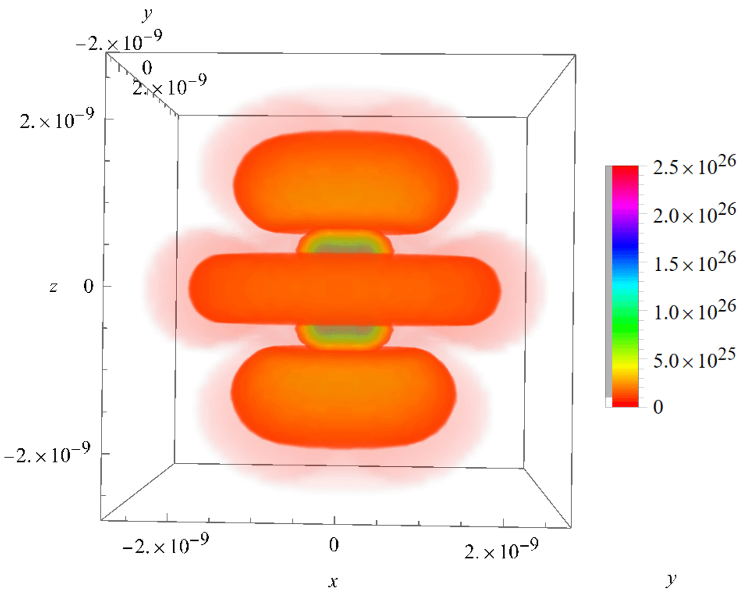
The region $3\pi/2 < \phi < 2\pi$ is excluded.

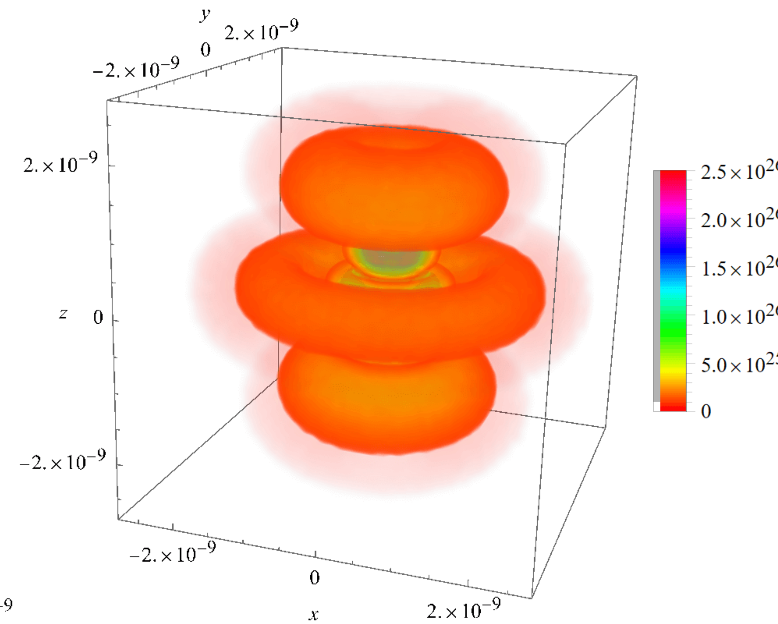
$$|\Psi_{530}(r,\theta,\phi,t)|^2$$

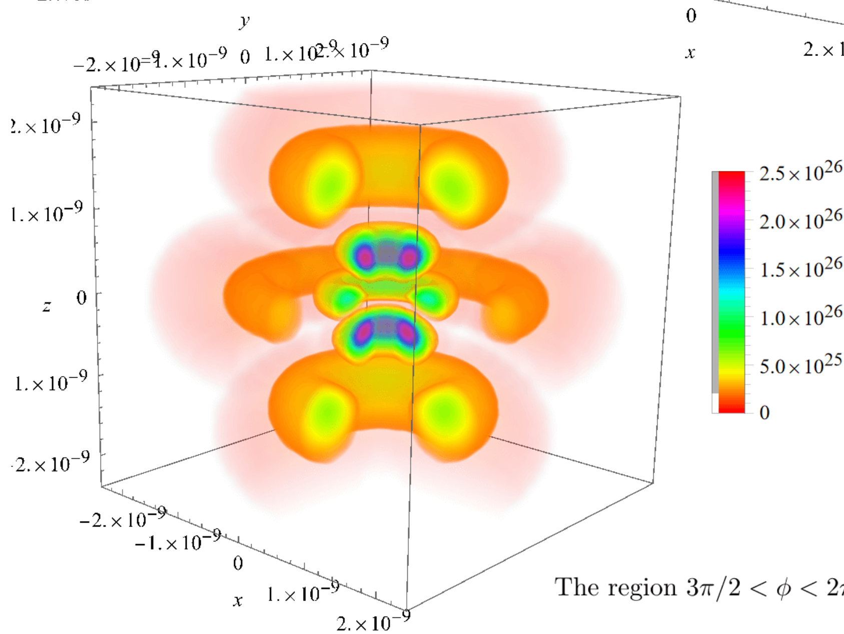


The region $3\pi/2 < \phi < 2\pi$ is excluded.

$$|\Psi_{53\pm1}(r, \theta, \phi, t)|^2$$

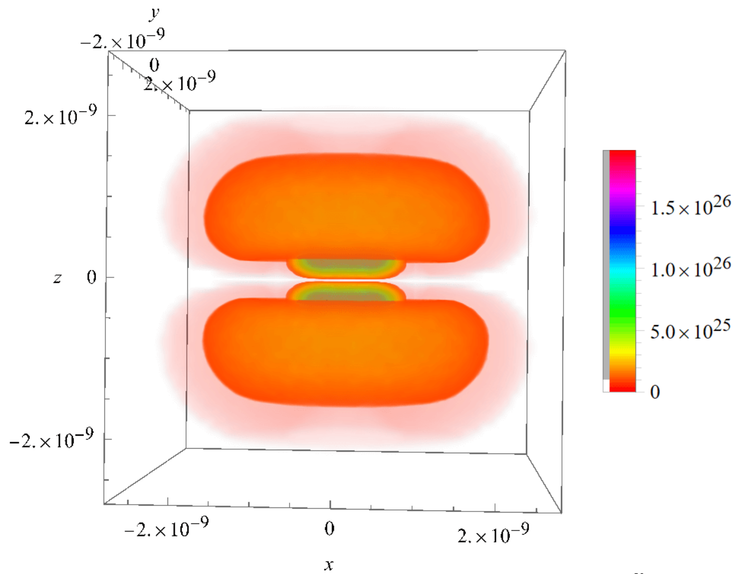


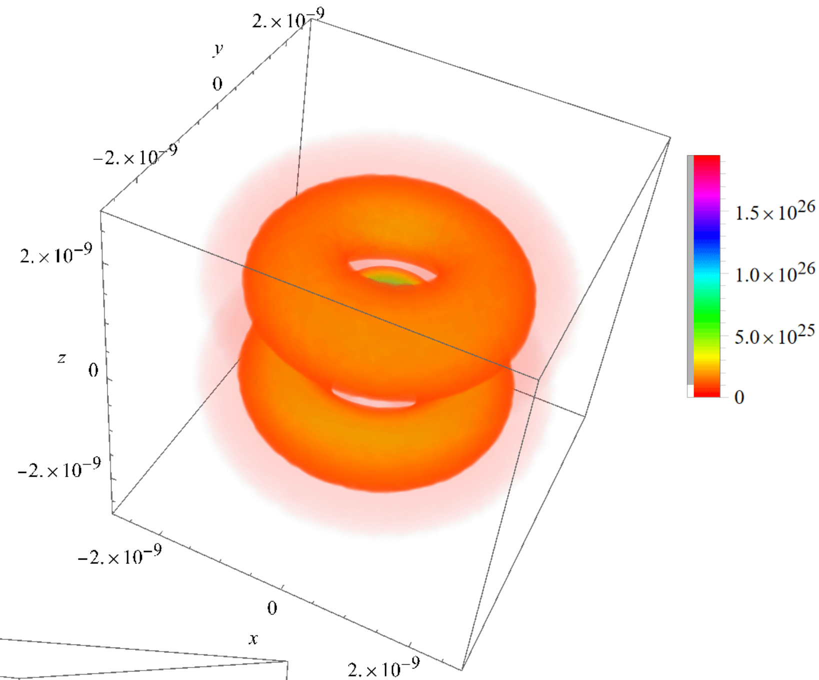


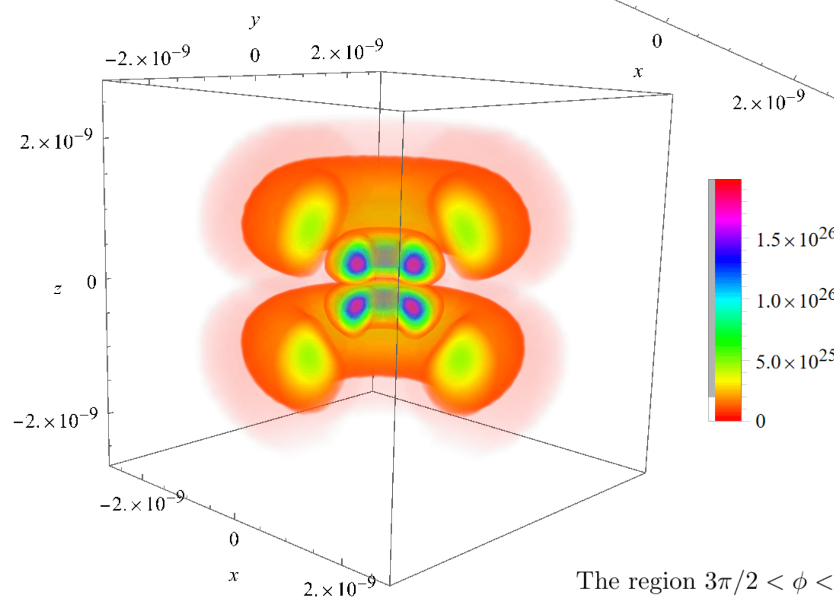


The region $3\pi/2 < \phi < 2\pi$ is excluded.

$$|\Psi_{53\pm2}(r,\theta,\phi,t)|^2$$

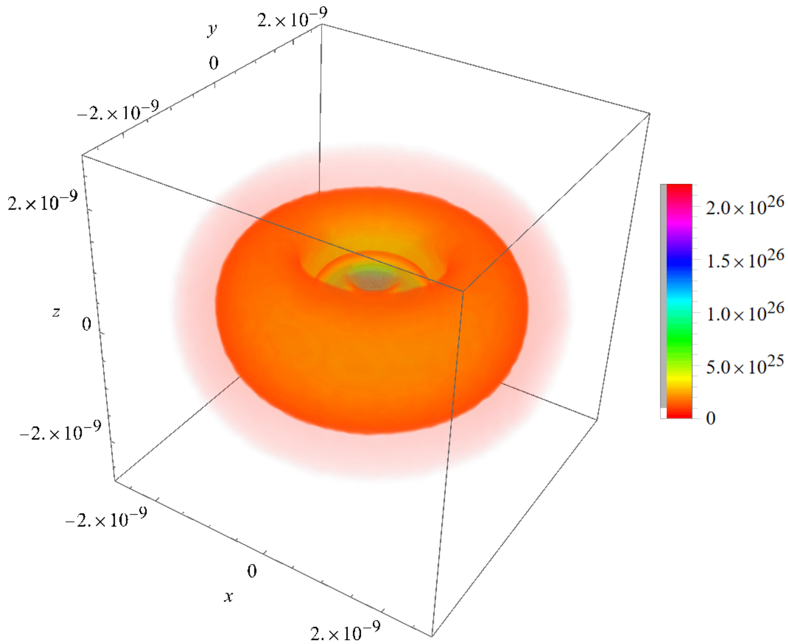


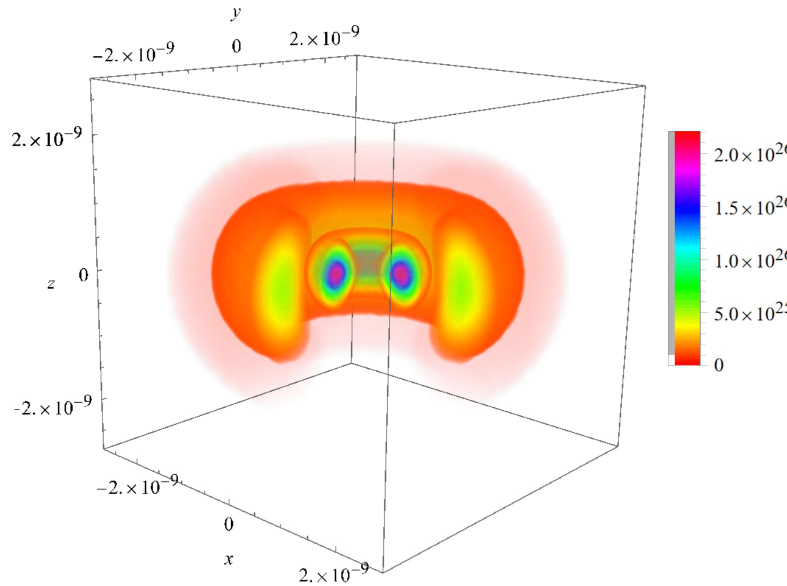




The region $3\pi/2 < \phi < 2\pi$ is excluded.

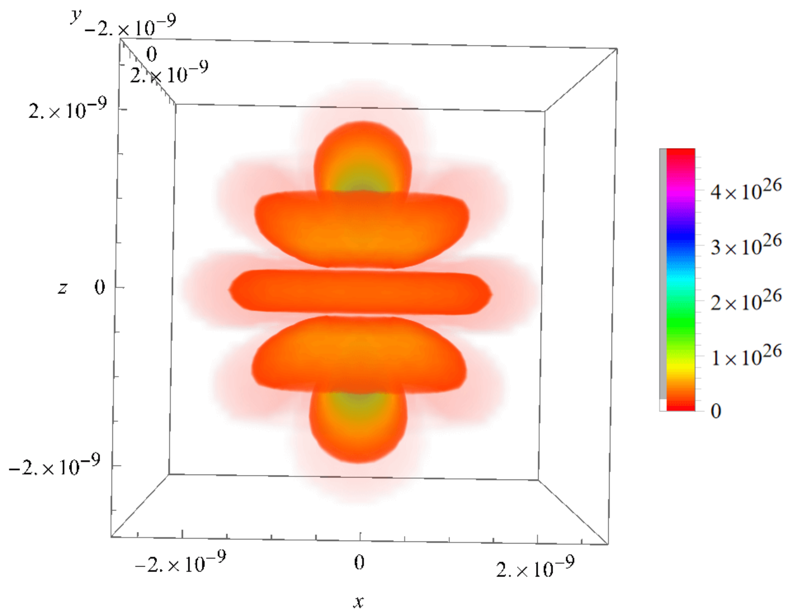
$$|\Psi_{53\pm3}(r,\theta,\phi,t)|^2$$

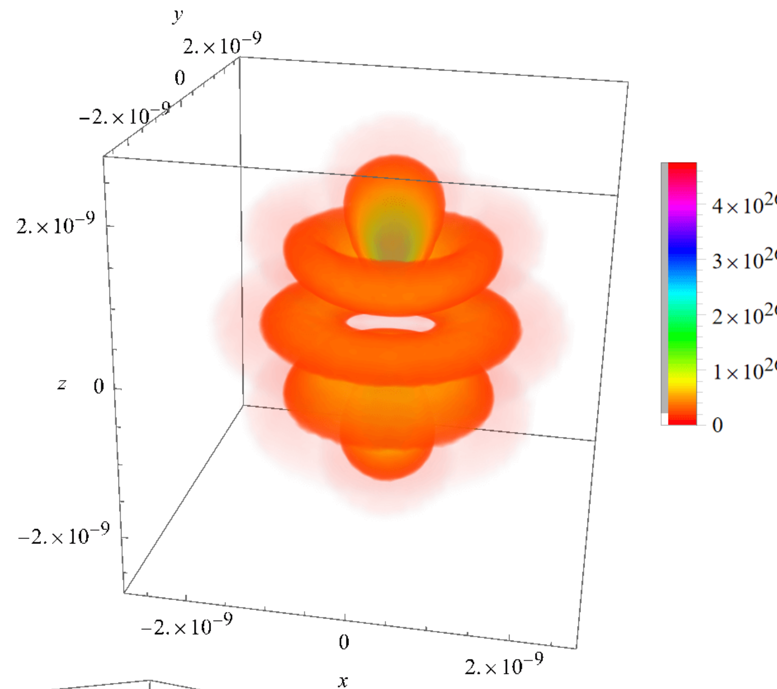


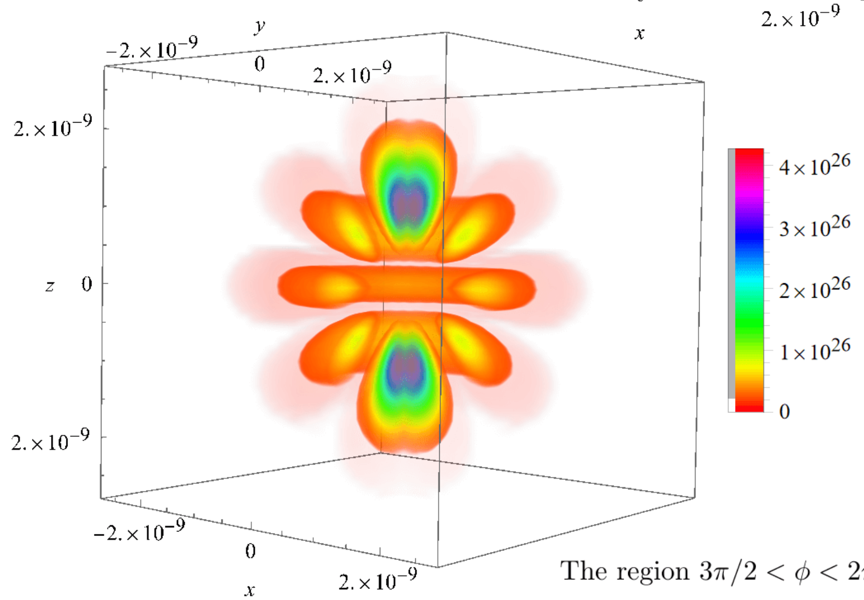


The region $3\pi/2 < \phi < 2\pi$ is excluded.

$$|\Psi_{540}(r, \theta, \phi, t)|^2$$

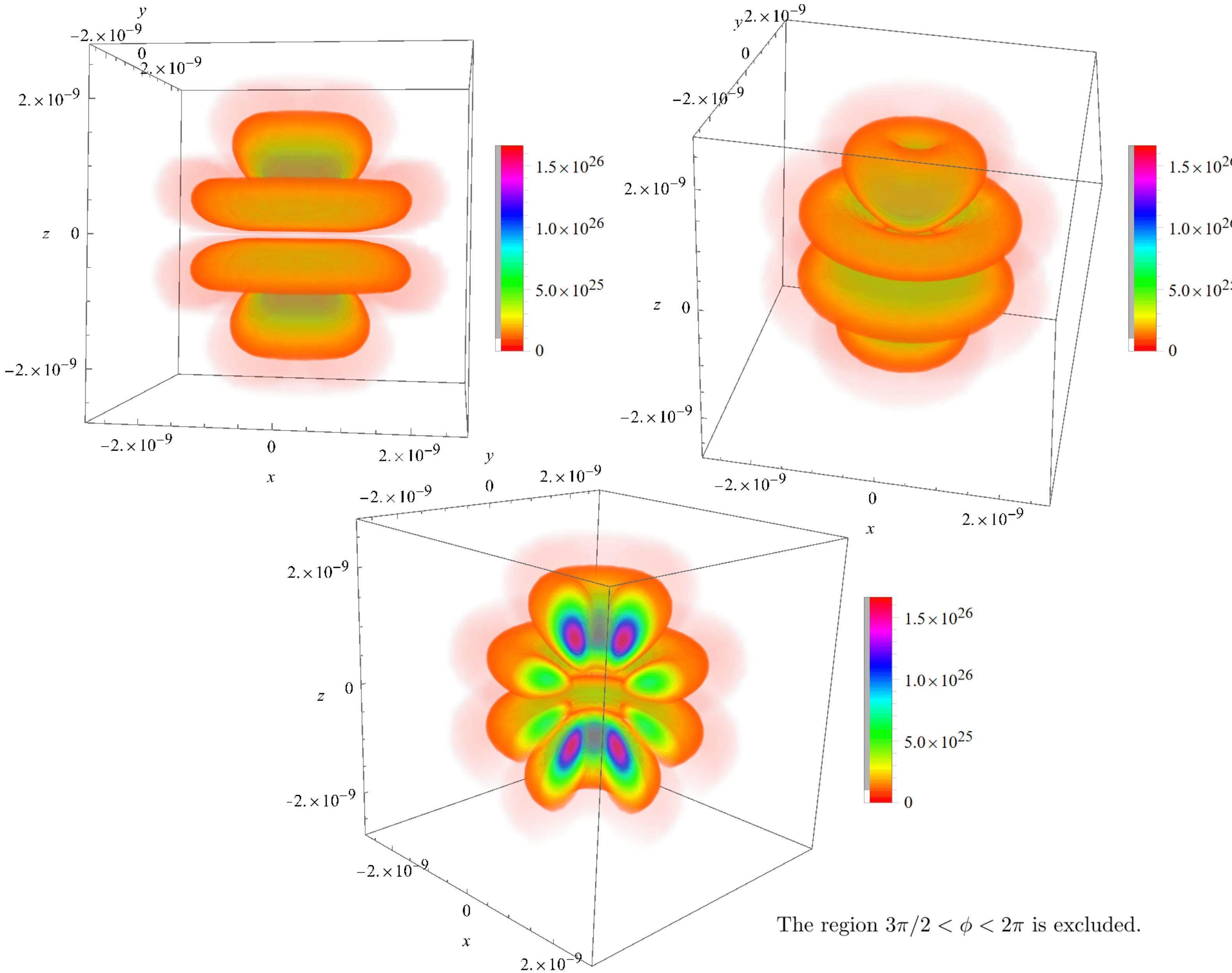






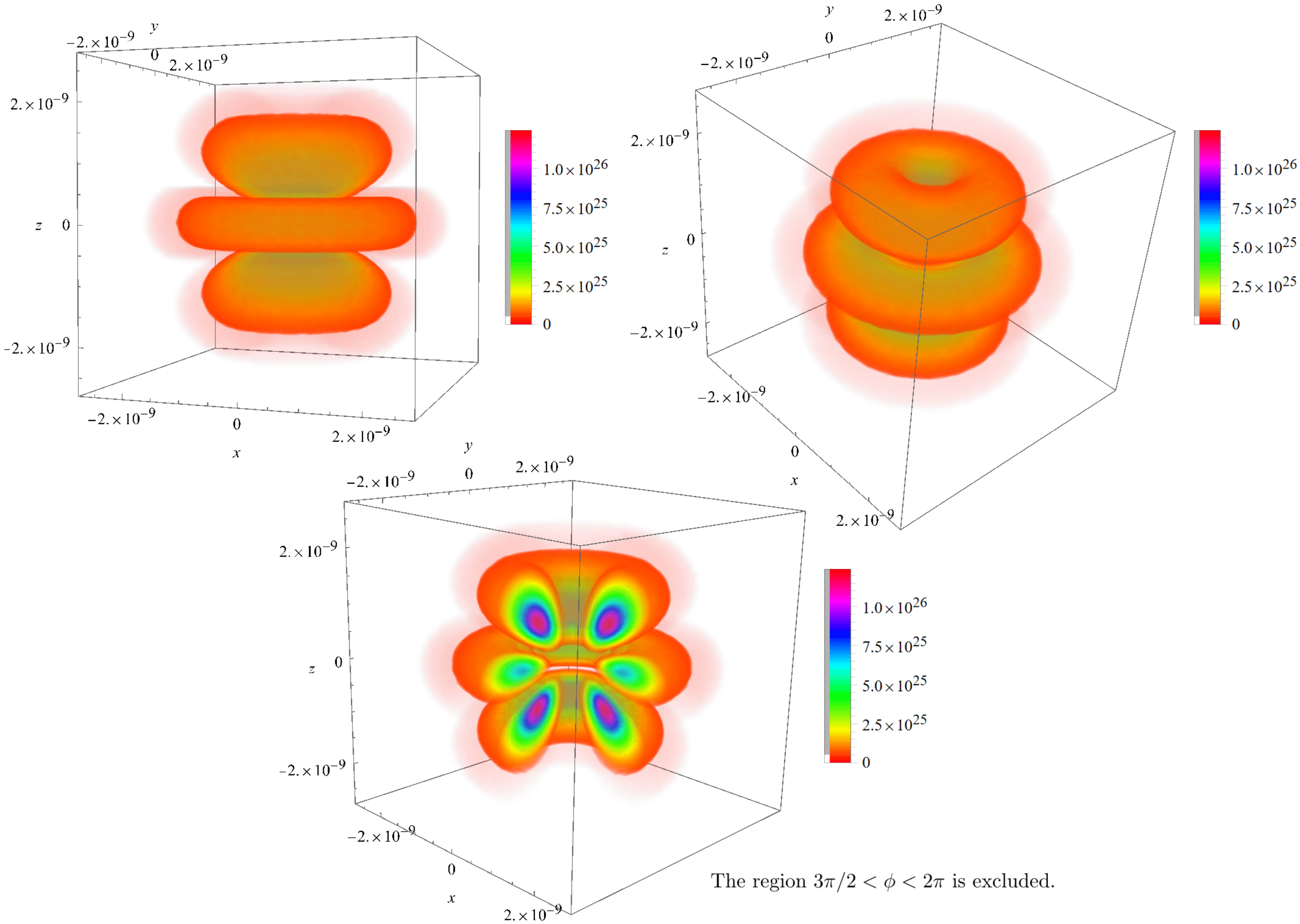
The region $3\pi/2 < \phi < 2\pi$ is excluded.

$$\left|\Psi_{54\pm1}(r,\theta,\phi,t)\right|^2$$



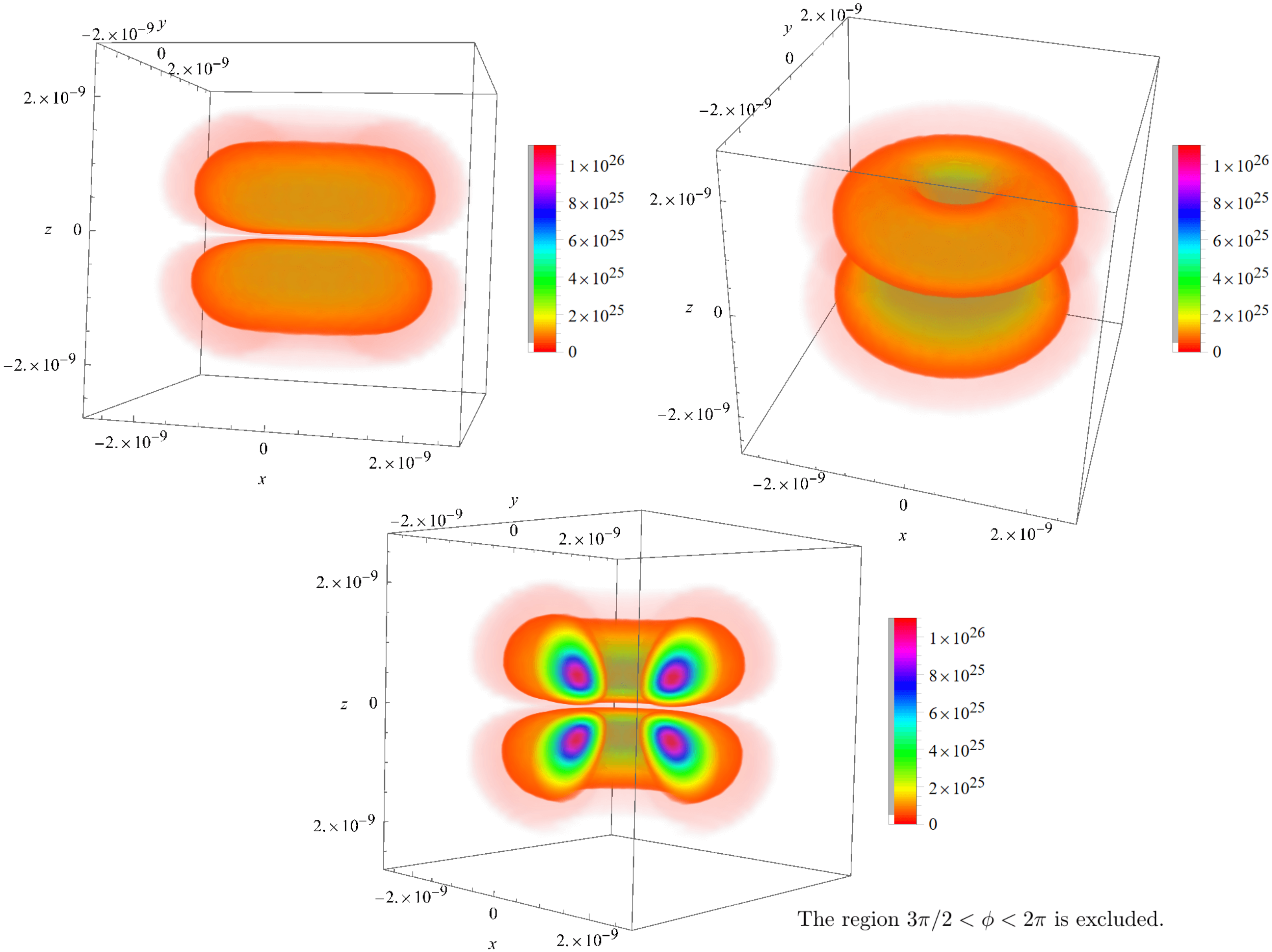
The region $3\pi/2 < \phi < 2\pi$ is excluded.

$$|\Psi_{54\pm2}(r,\theta,\phi,t)|^2$$



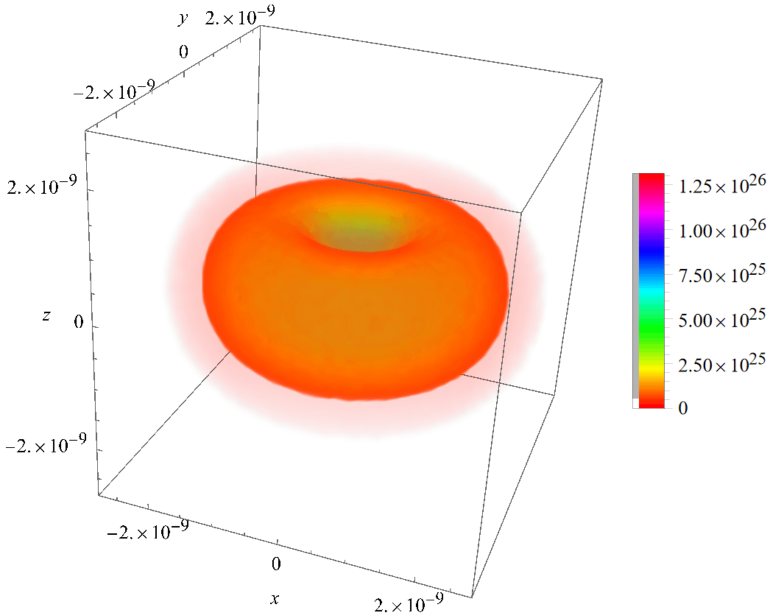
The region $3\pi/2 < \phi < 2\pi$ is excluded.

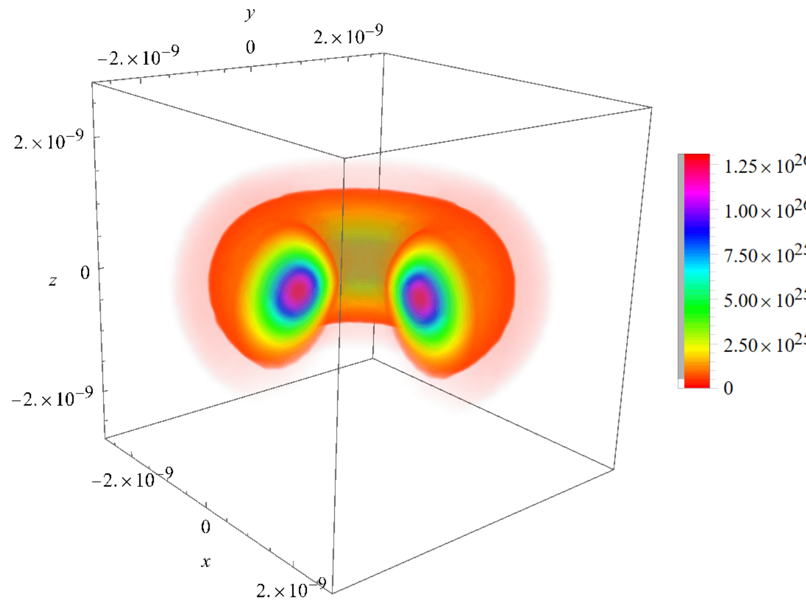
$$|\Psi_{54\pm3}(r,\theta,\phi,t)|^2$$



The region $3\pi/2 < \phi < 2\pi$ is excluded.

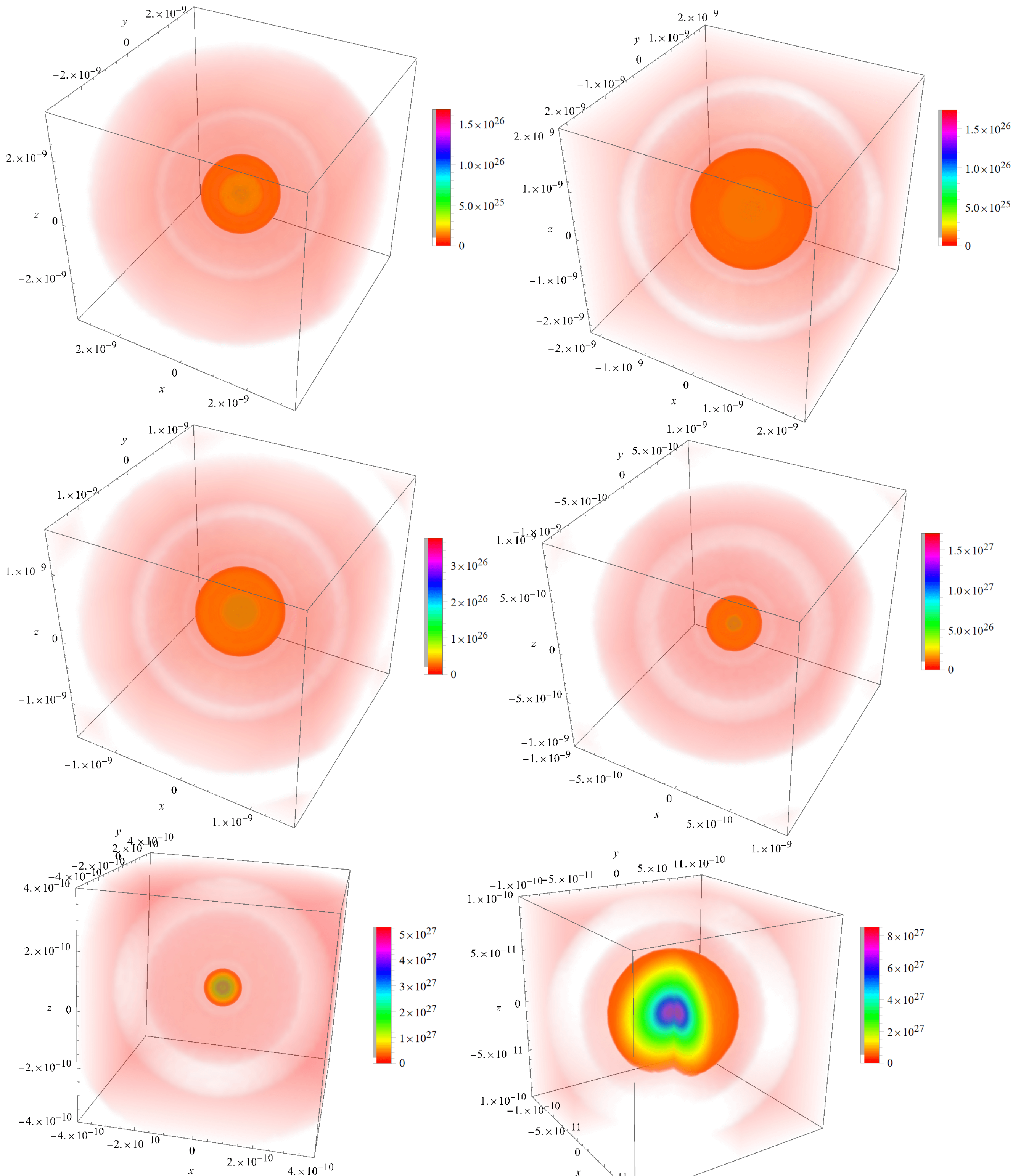
$$|\Psi_{54\pm4}(r,\theta,\phi,t)|^2$$





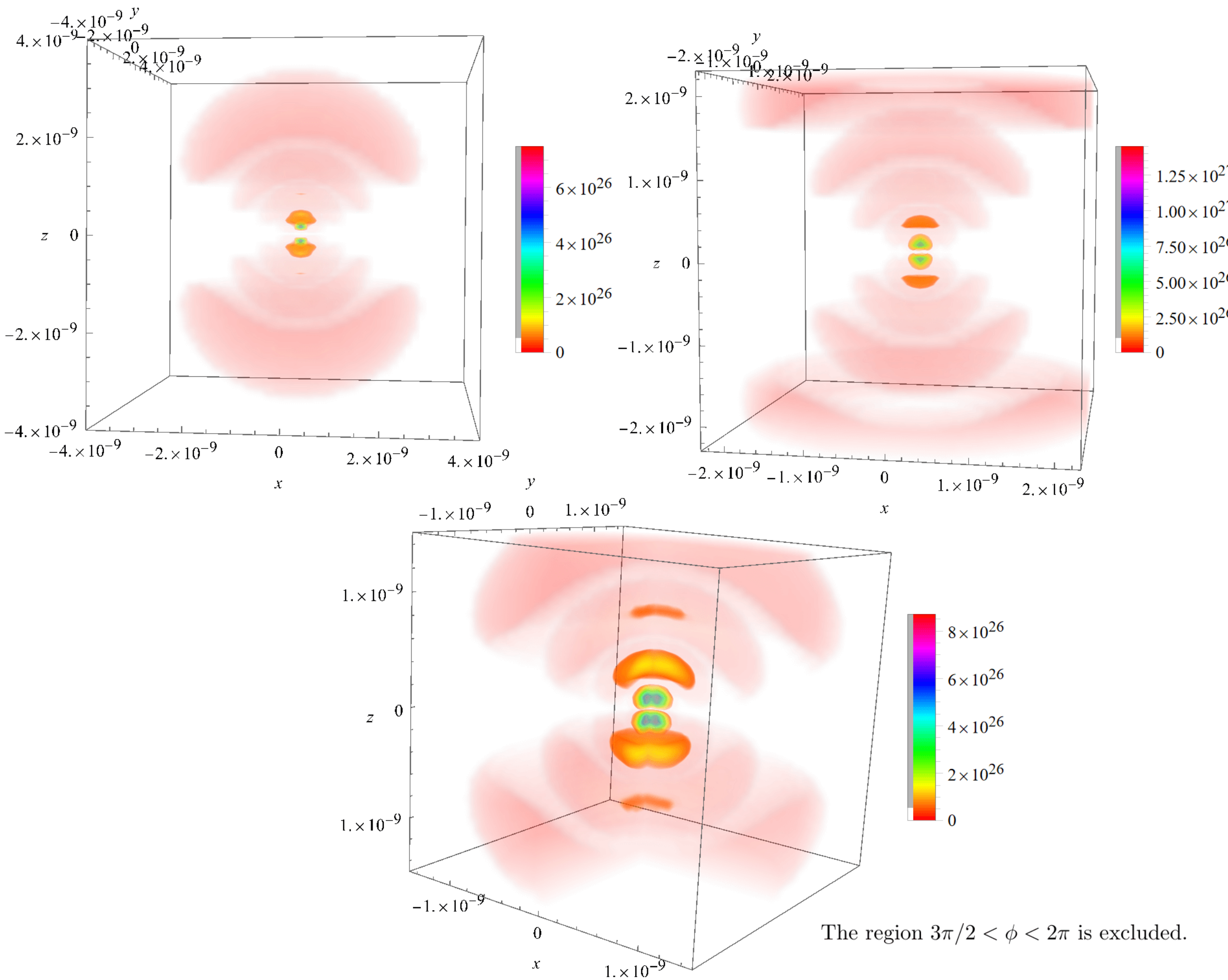
The region $3\pi/2 < \phi < 2\pi$ is excluded.

$|\Psi_{600}(r, \theta, \phi, t)|^2$



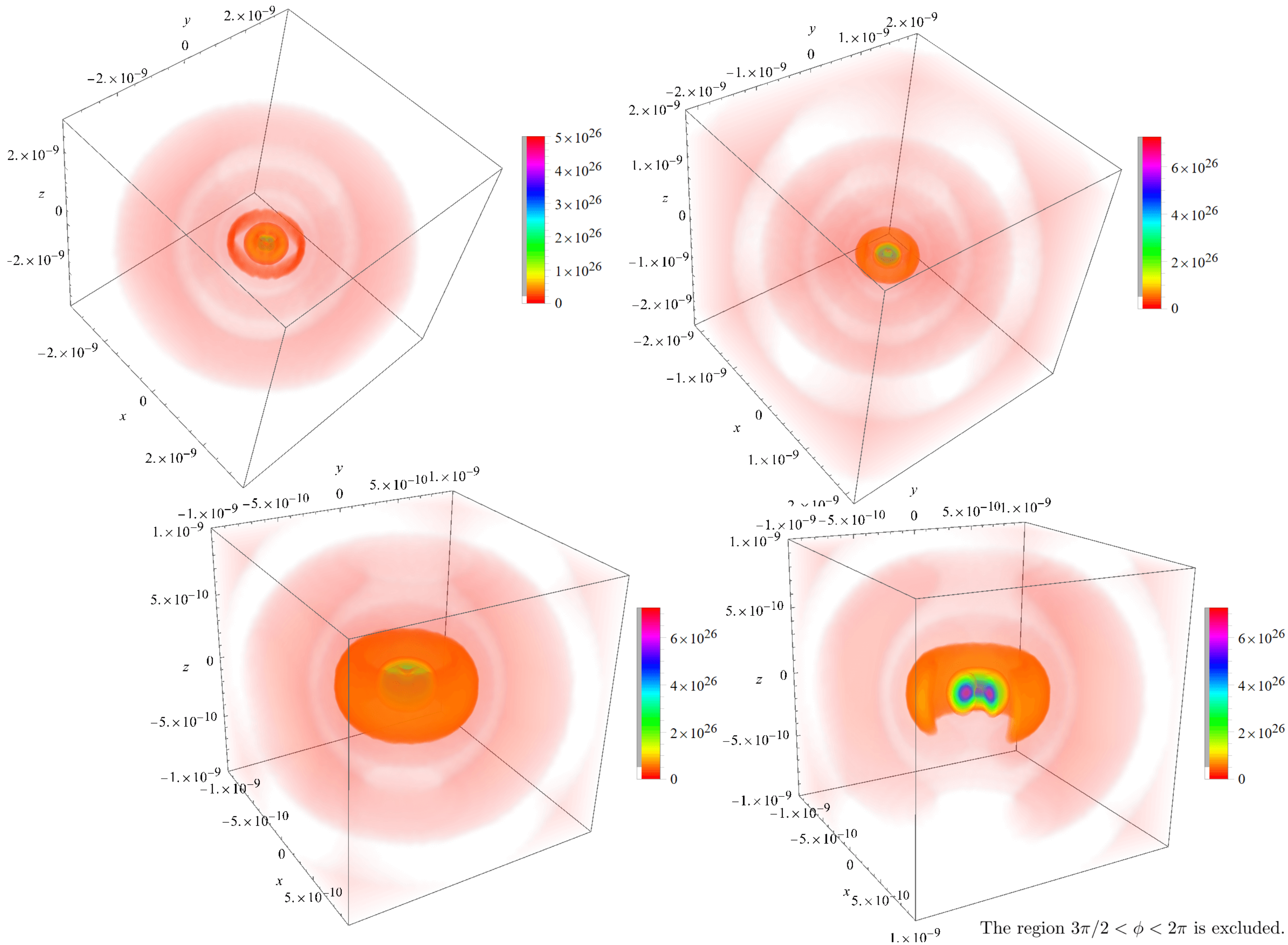
The region $3\pi/2 < \phi < 2\pi$ is excluded.

$$|\Psi_{610}(r,\theta,\phi,t)|^2$$



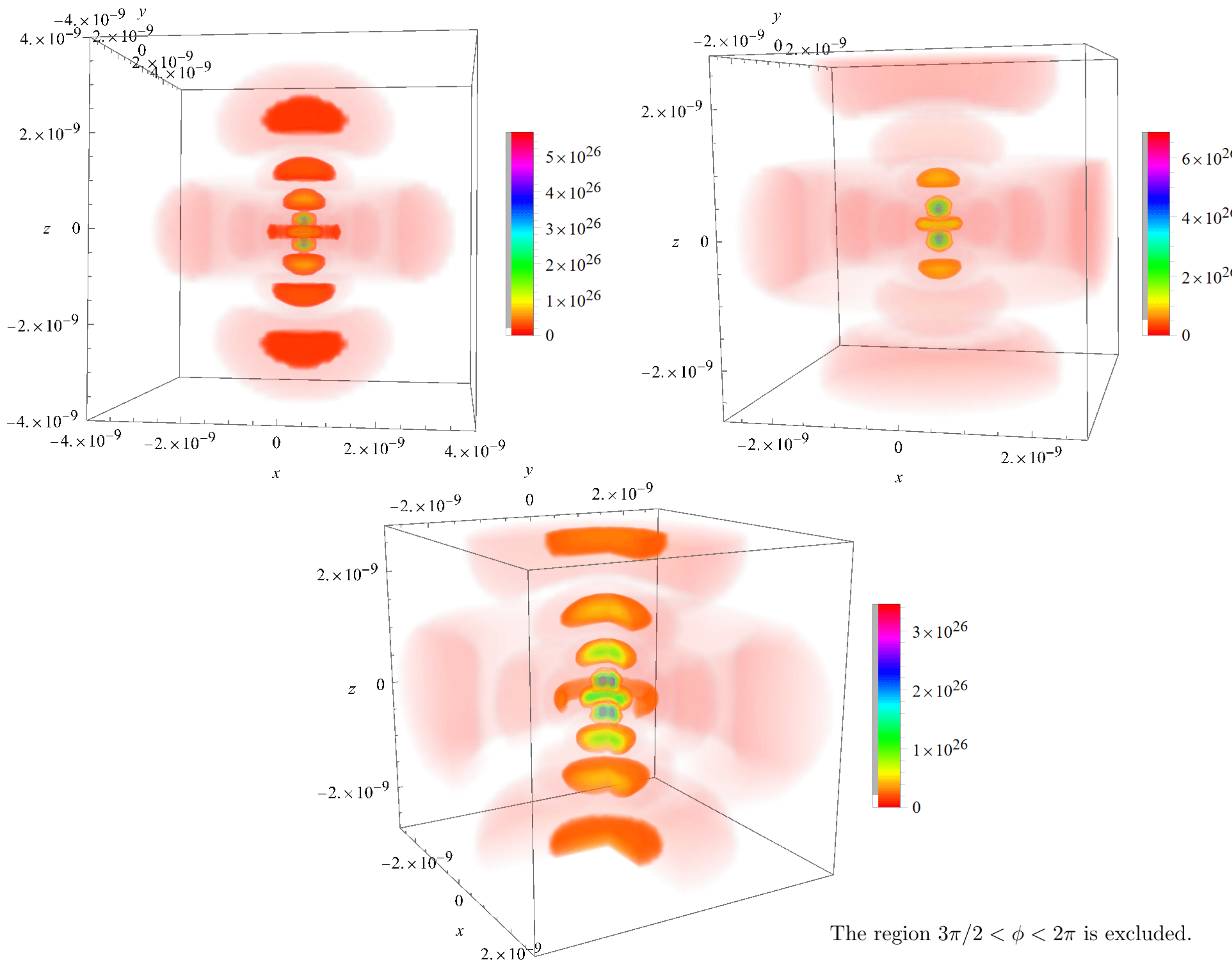
The region $3\pi/2 < \phi < 2\pi$ is excluded.

$|\Psi_{61\pm1}(r,\theta,\phi,t)|^2$



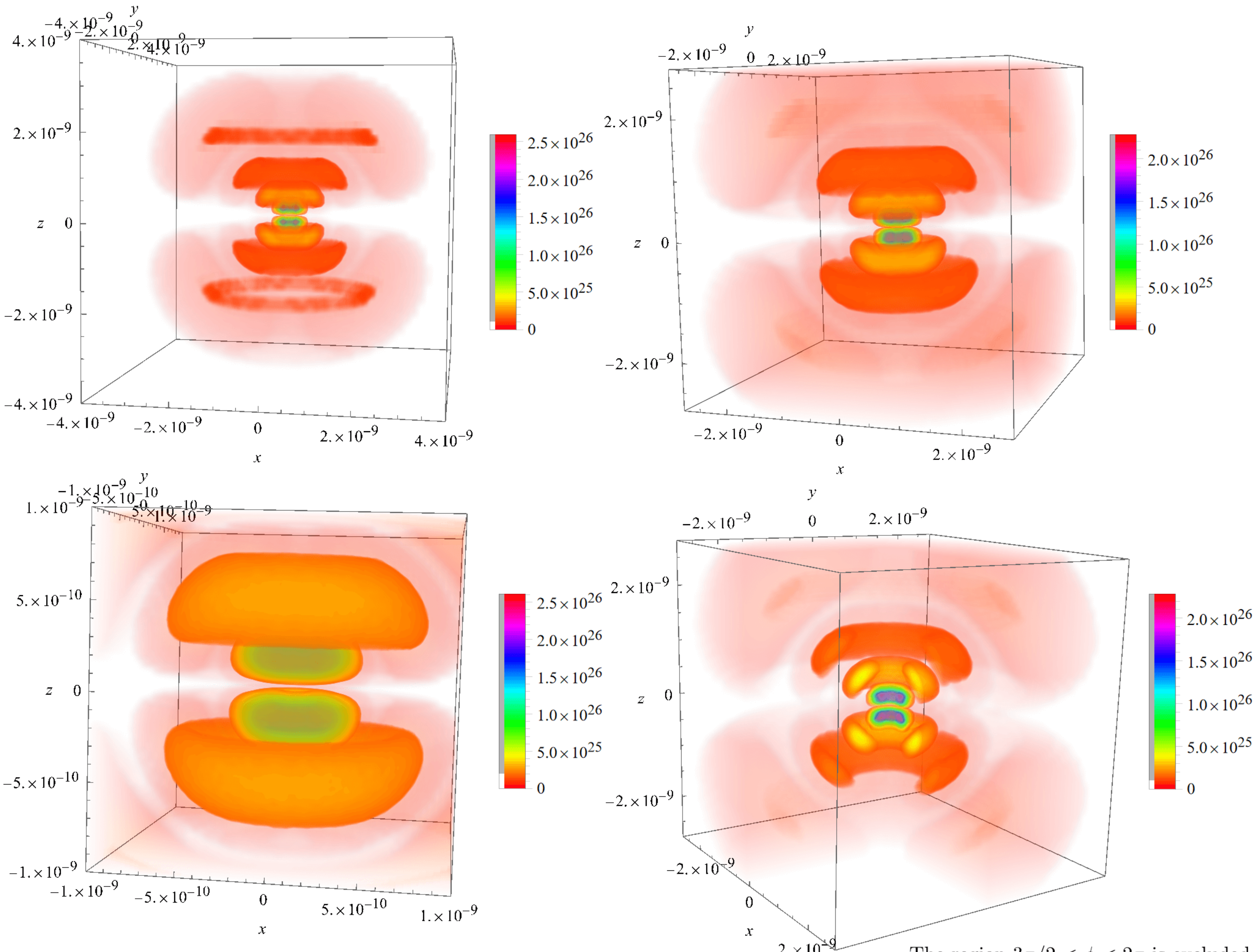
The region $3\pi/2 < \phi < 2\pi$ is excluded.

$$|\Psi_{620}(r, \theta, \phi, t)|^2$$



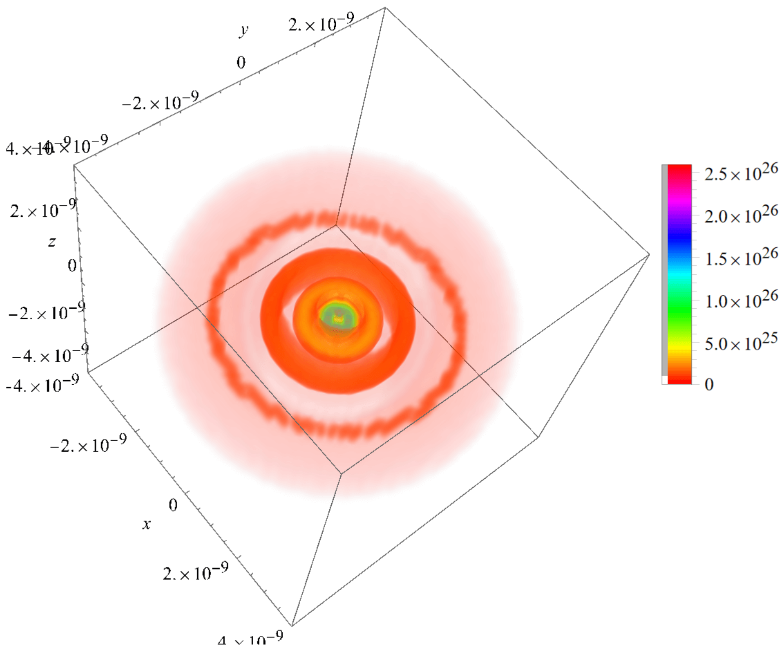
The region $3\pi/2 < \phi < 2\pi$ is excluded.

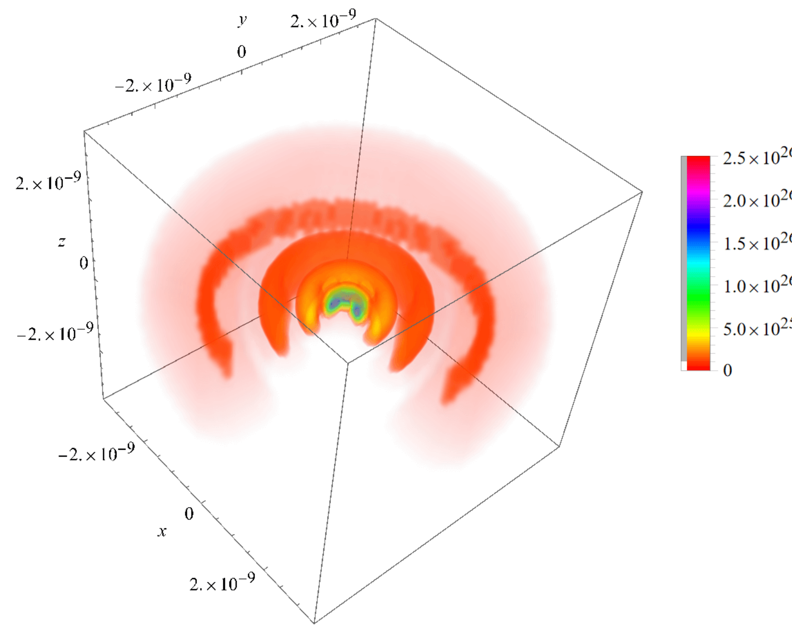
$$|\Psi_{62\pm1}(r,\theta,\phi,t)|^2$$



The region $3\pi/2 < \phi < 2\pi$ is excluded.

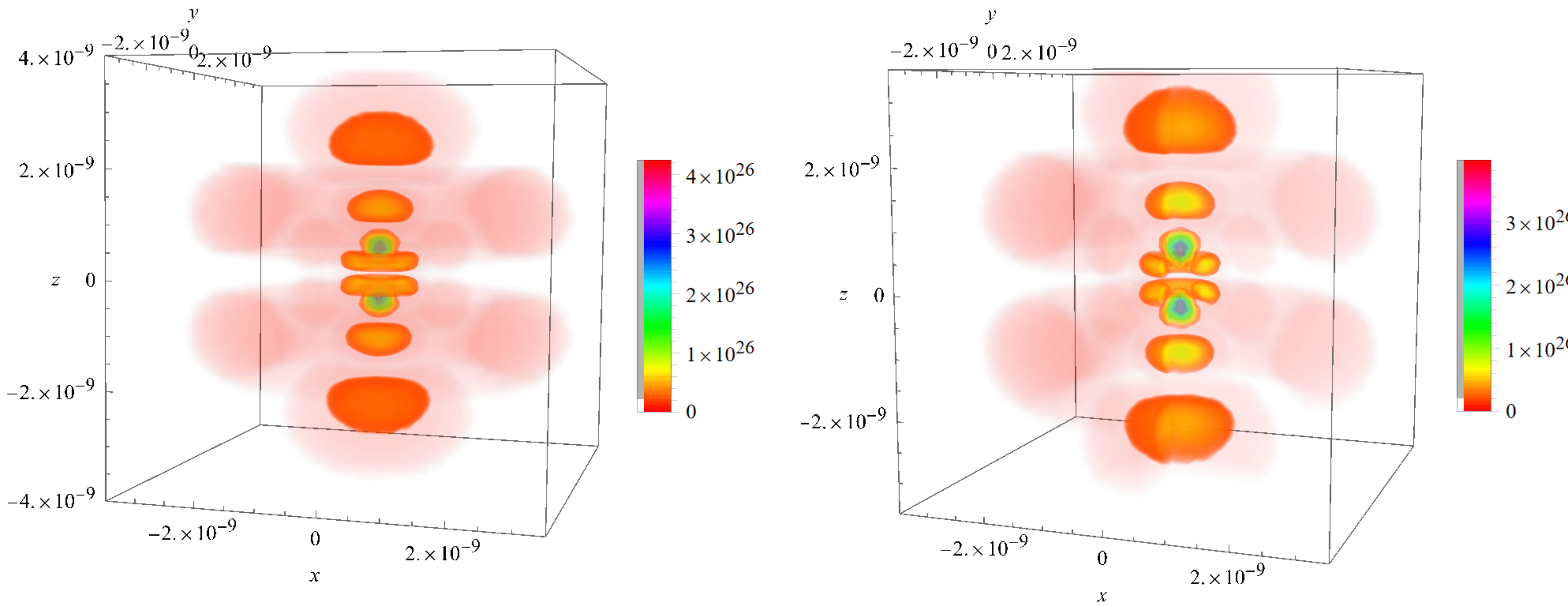
$|\Psi_{62\pm2}(r,\,\theta,\,\phi,\,t)|^2$





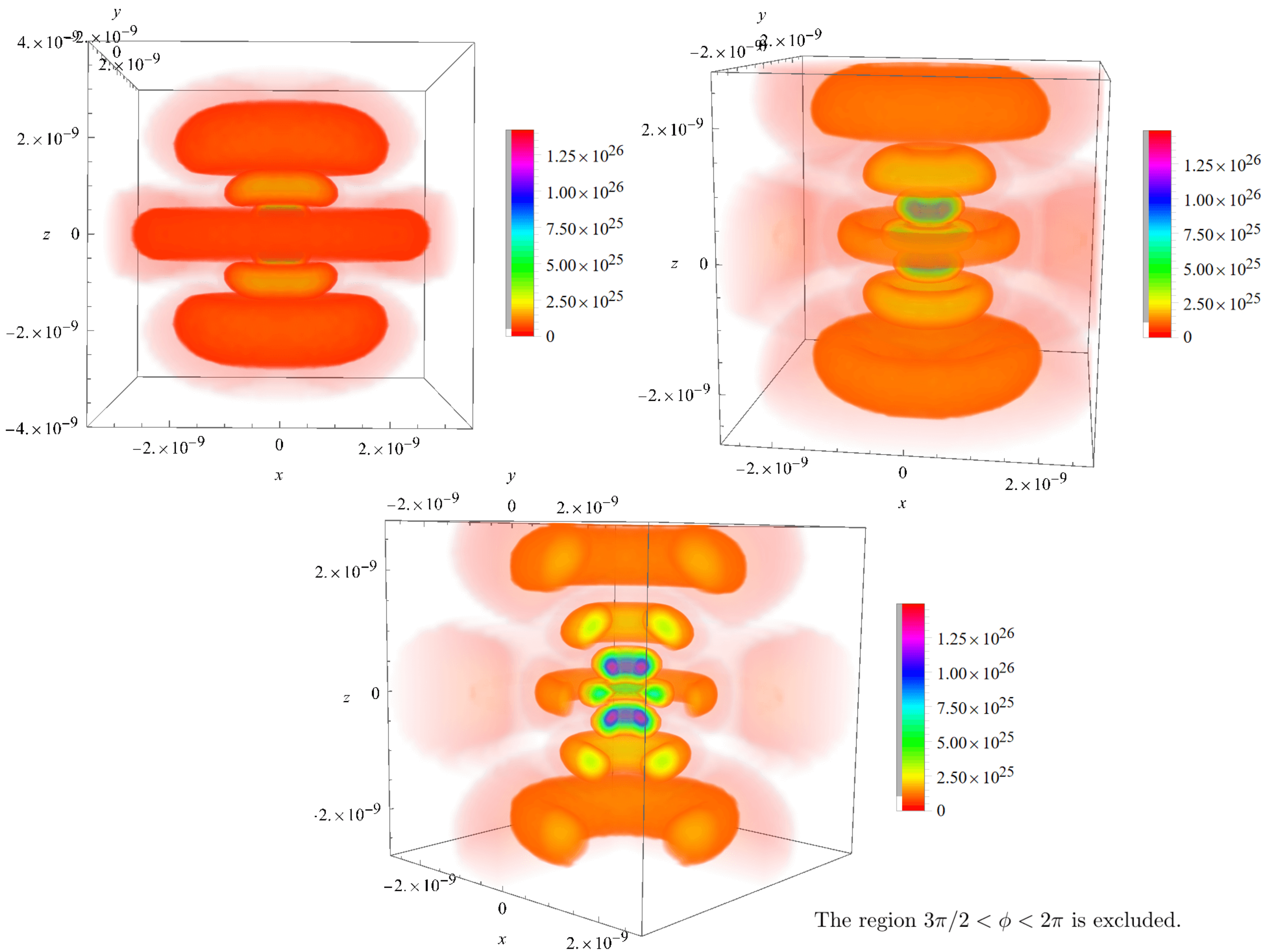
The region $3\pi/2 < \phi < 2\pi$ is excluded.

$$|\Psi_{630}(r, \theta, \phi, t)|^2$$



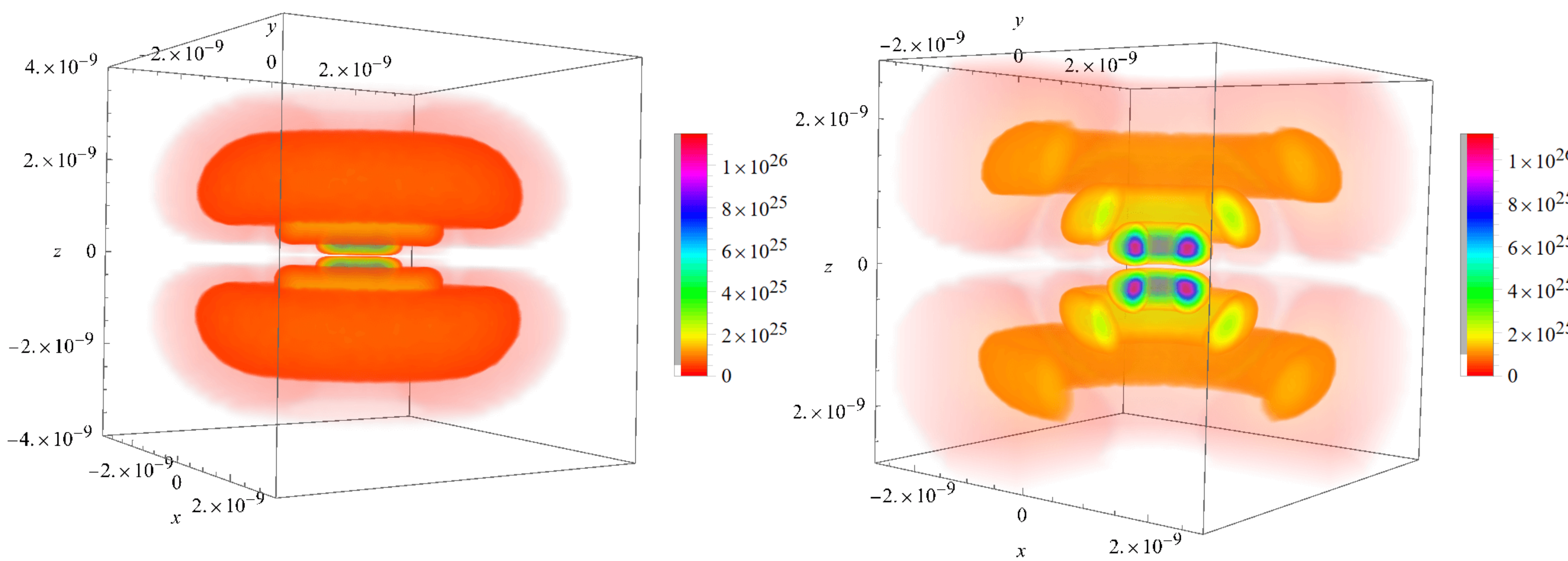
The region $3\pi/2 < \phi < 2\pi$ is excluded.

$$|\Psi_{63\pm1}(r, \theta, \phi, t)|^2$$



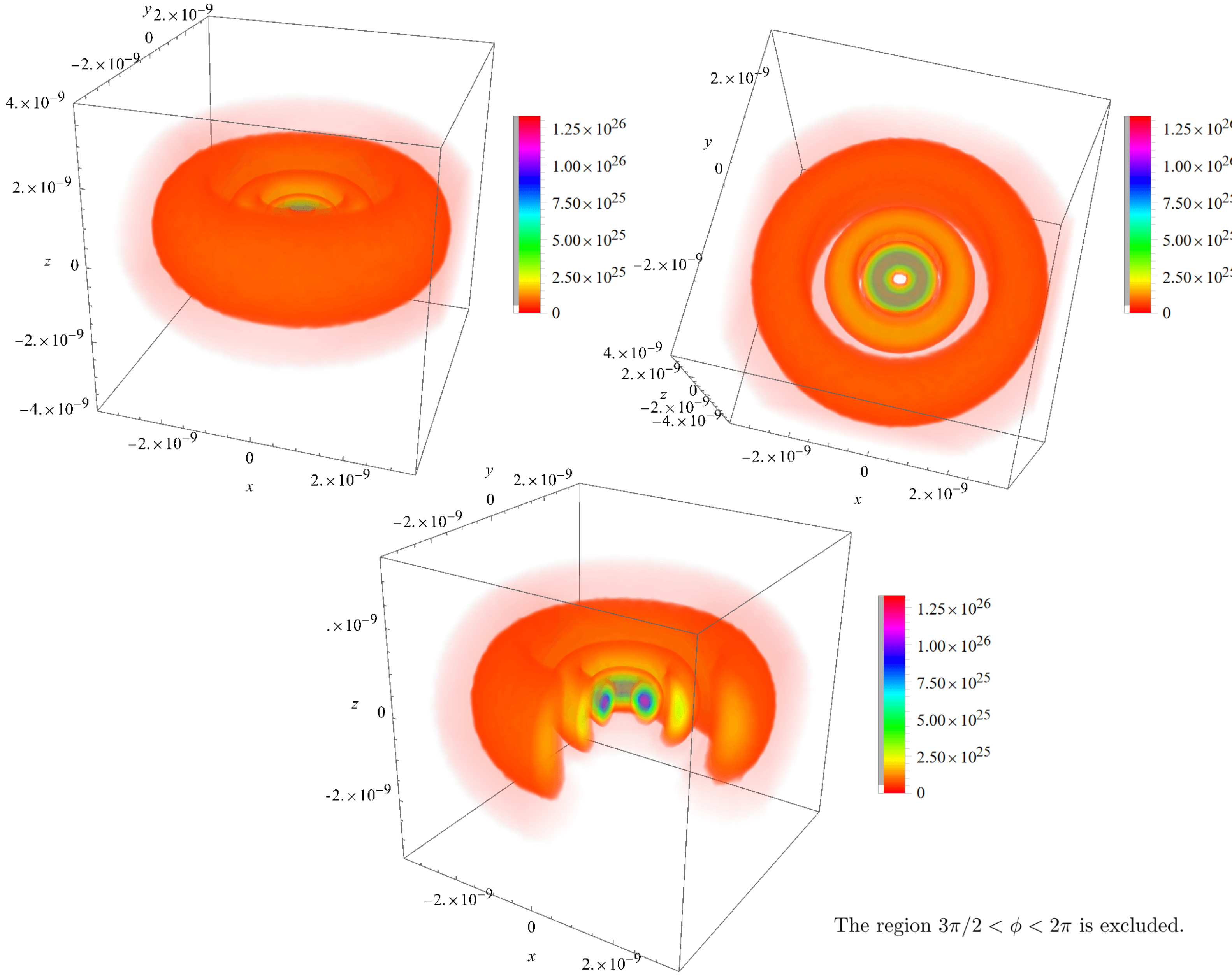
The region $3\pi/2 < \phi < 2\pi$ is excluded.

$$|\Psi_{63\pm2}(r, \theta, \phi, t)|^2$$



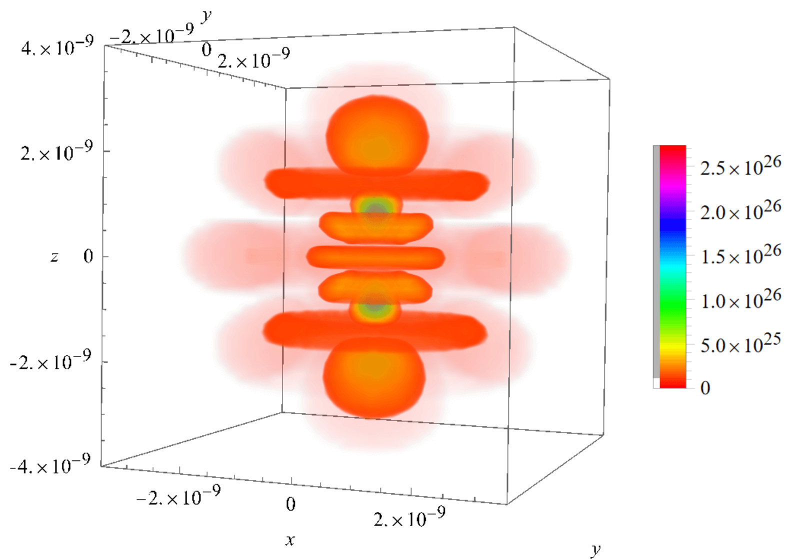
The region $3\pi/2 < \phi < 2\pi$ is excluded.

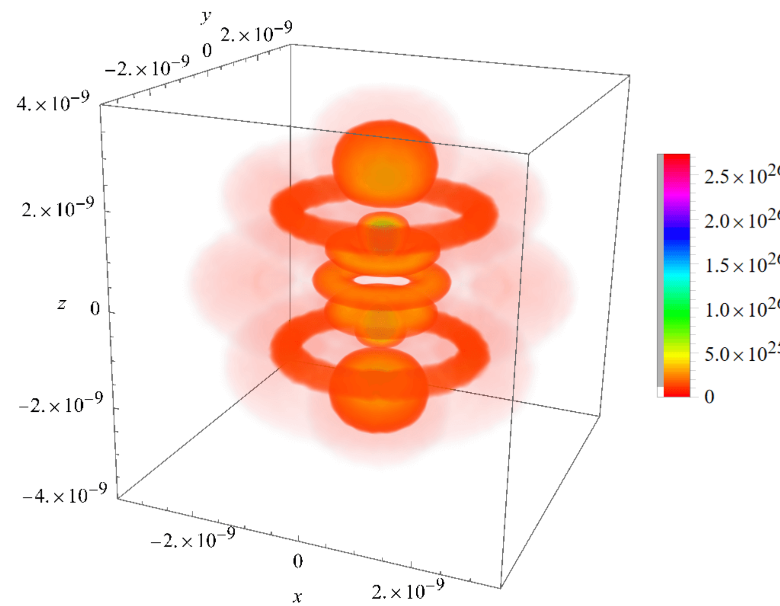
$$\left|\Psi_{63\pm3}(r,\,\theta,\,\phi,\,t)\right|^2$$

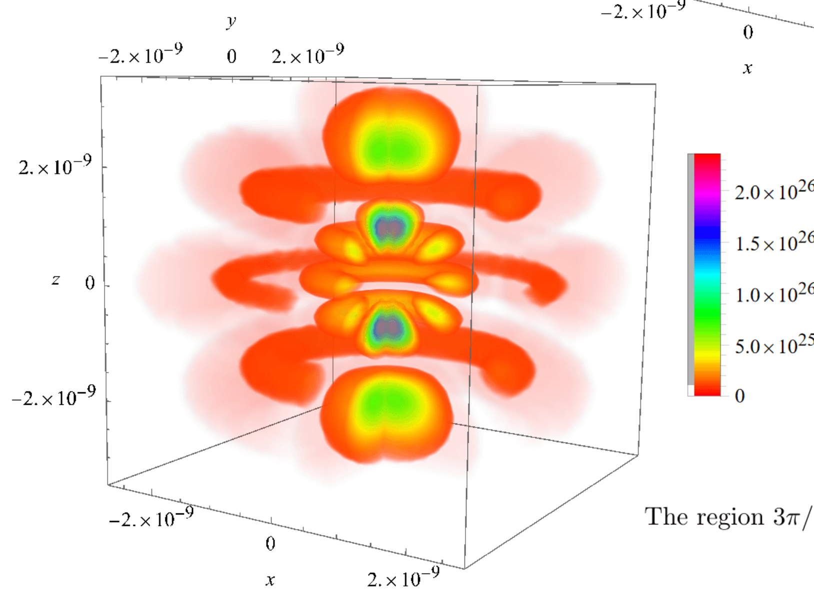


The region $3\pi/2 < \phi < 2\pi$ is excluded.

$$|\Psi_{640}(r, \theta, \phi, t)|^2$$

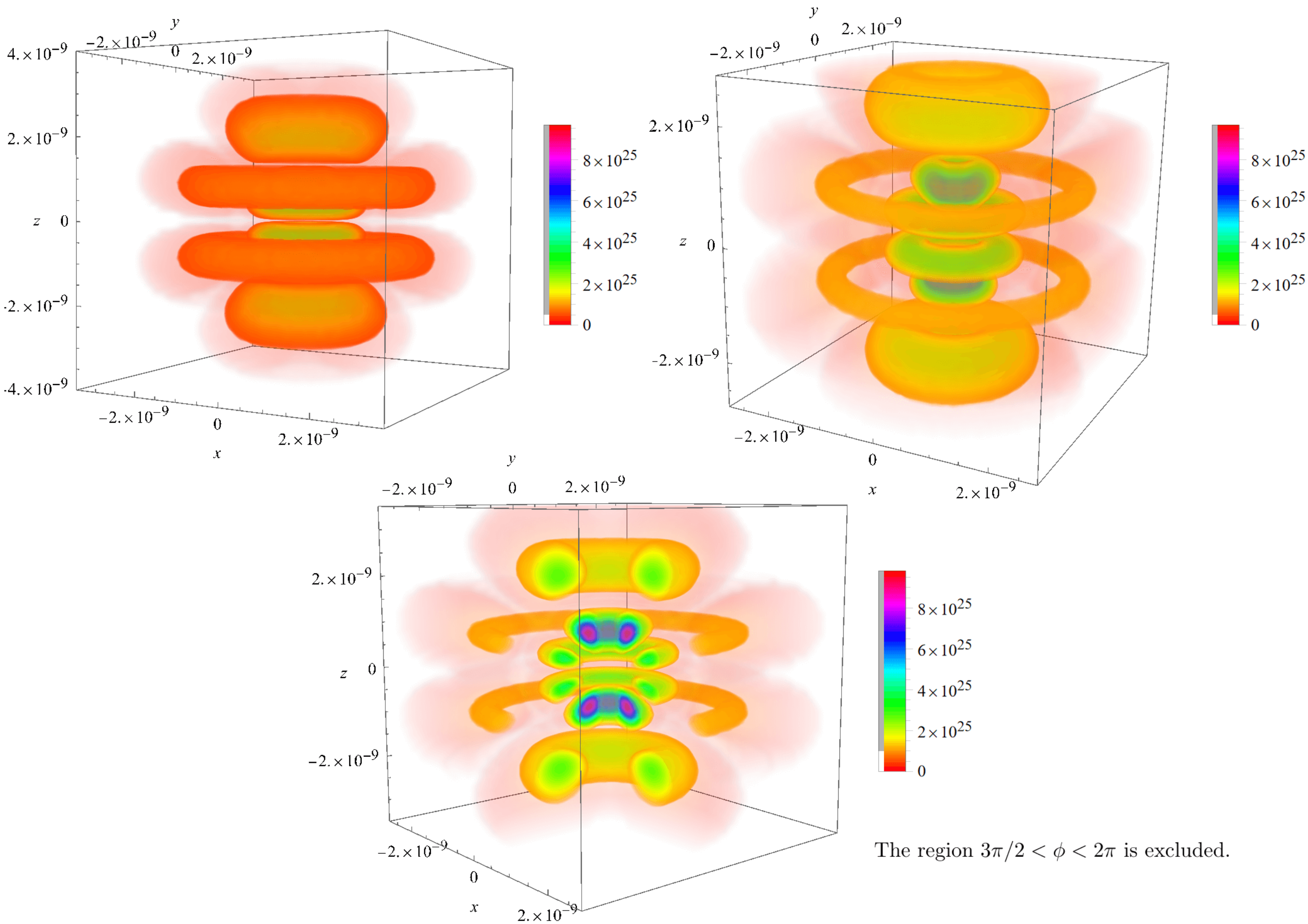






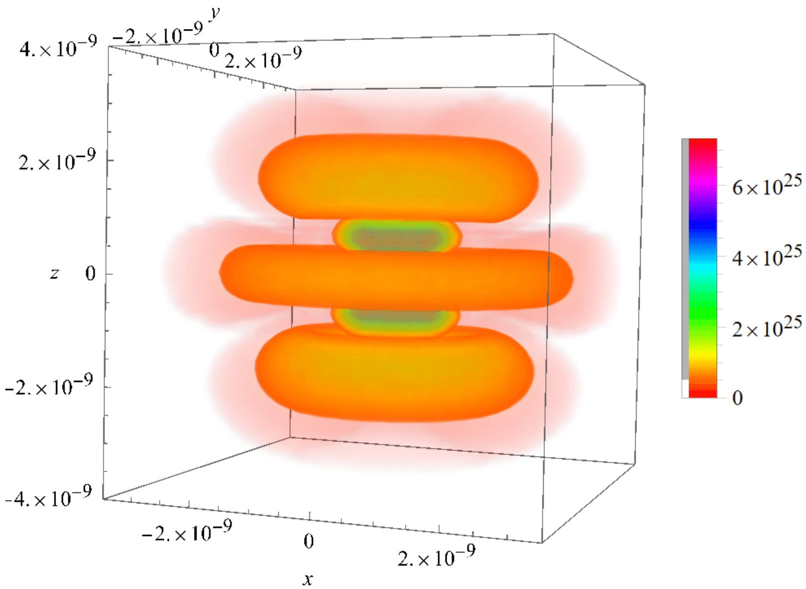
The region $3\pi/2 < \phi < 2\pi$ is excluded.

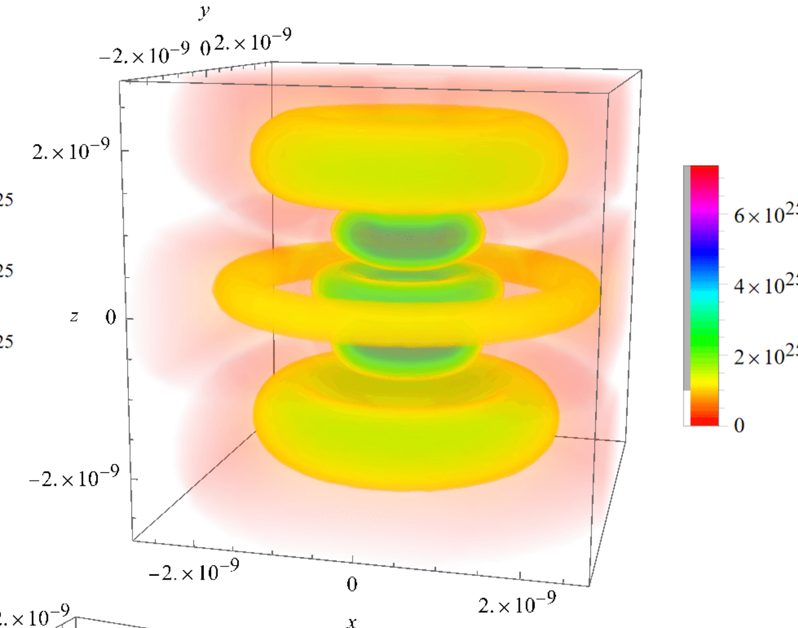
$$|\Psi_{64\pm1}(r,\theta,\phi,t)|^2$$

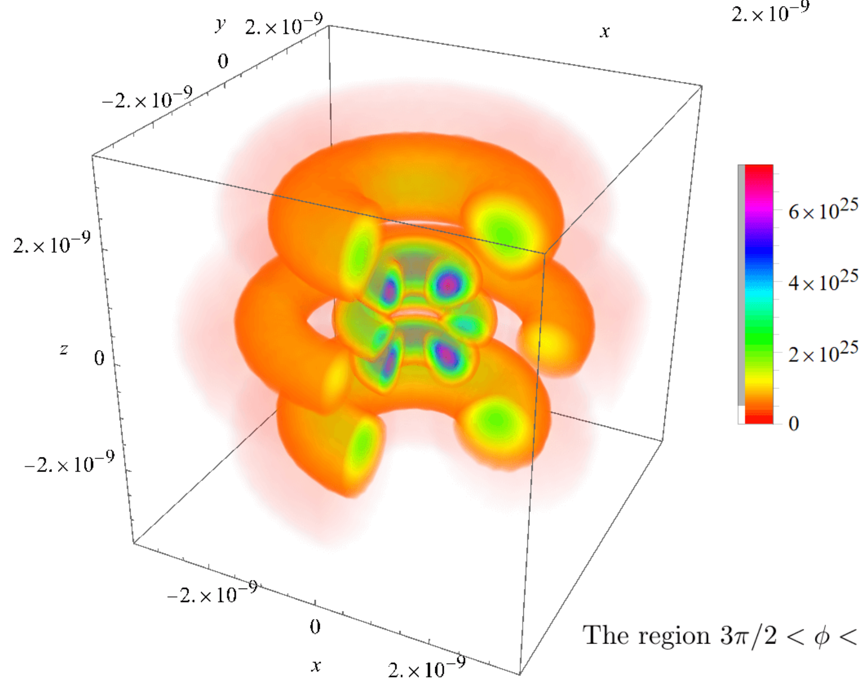


The region $3\pi/2 < \phi < 2\pi$ is excluded.

$$|\Psi_{64\pm2}(r, \theta, \phi, t)|^2$$

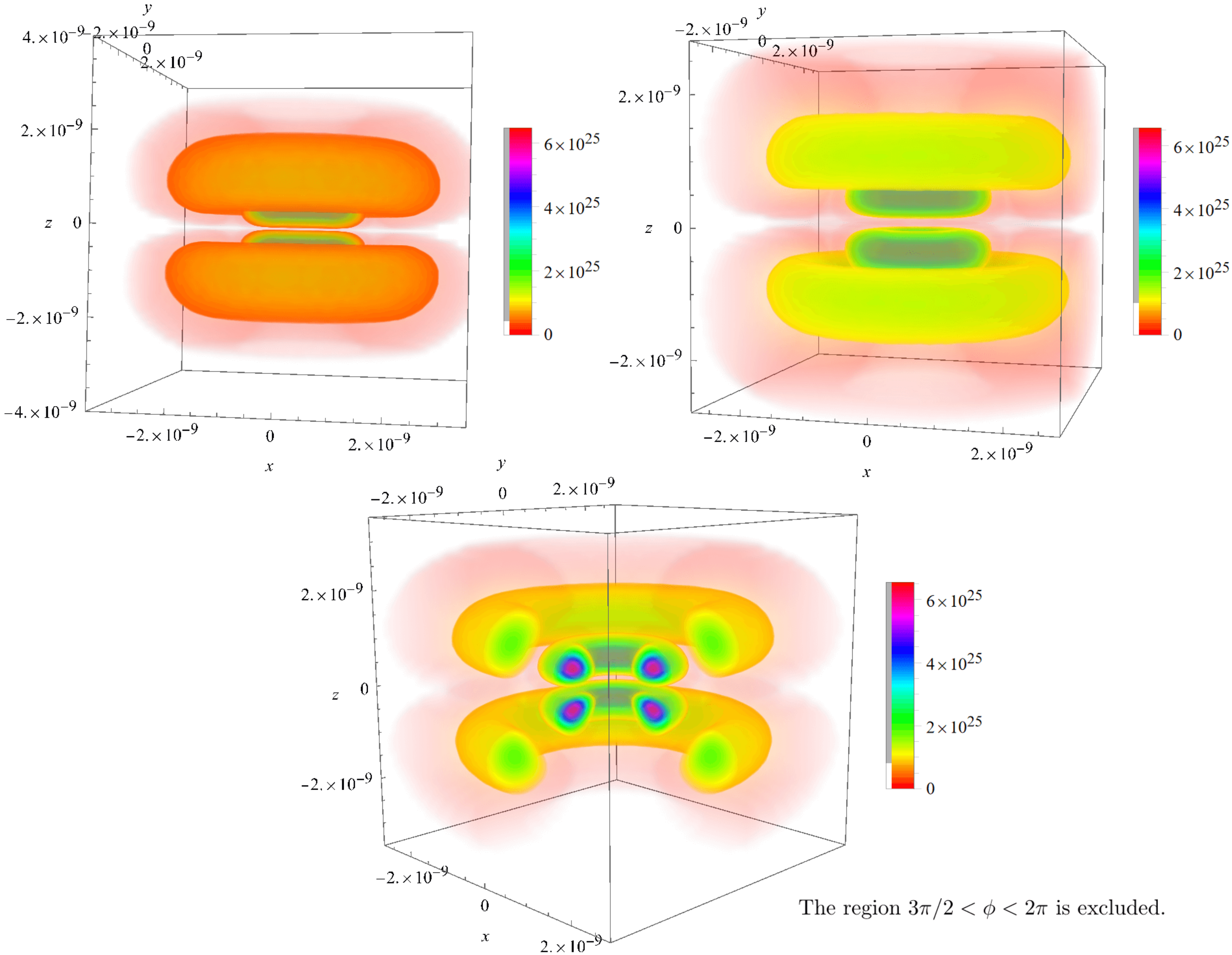






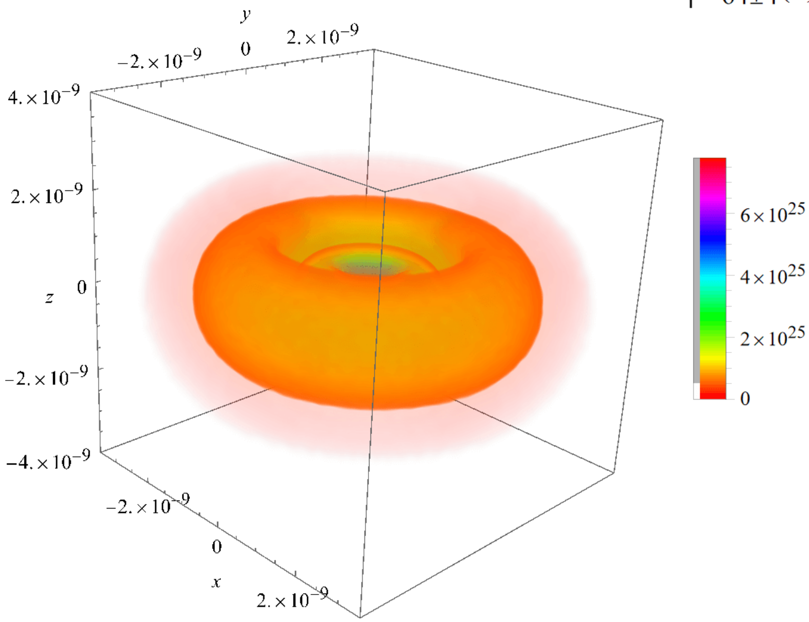
The region $3\pi/2 < \phi < 2\pi$ is excluded.

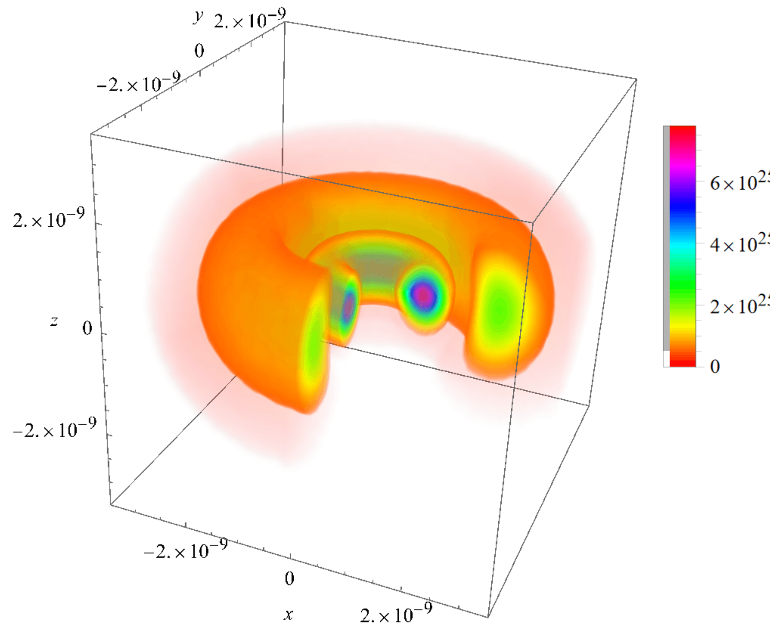
$$|\Psi_{64\pm3}(r, \theta, \phi, t)|^2$$



The region $3\pi/2 < \phi < 2\pi$ is excluded.

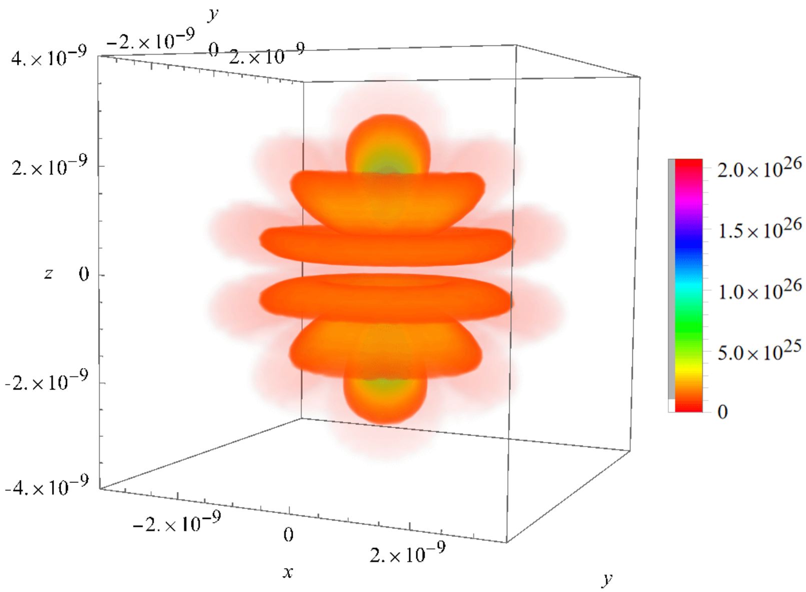
$$|\Psi_{64\pm4}(r,\theta,\phi,t)|^2$$

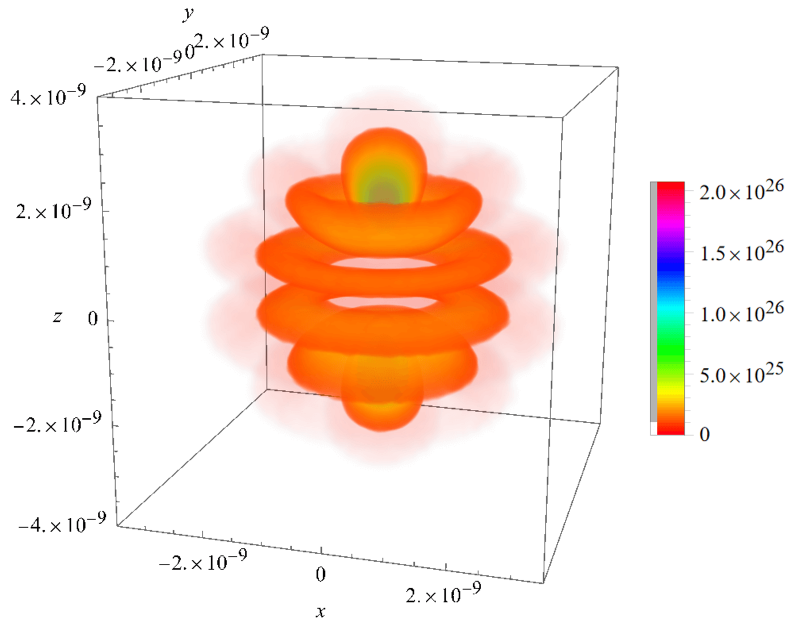


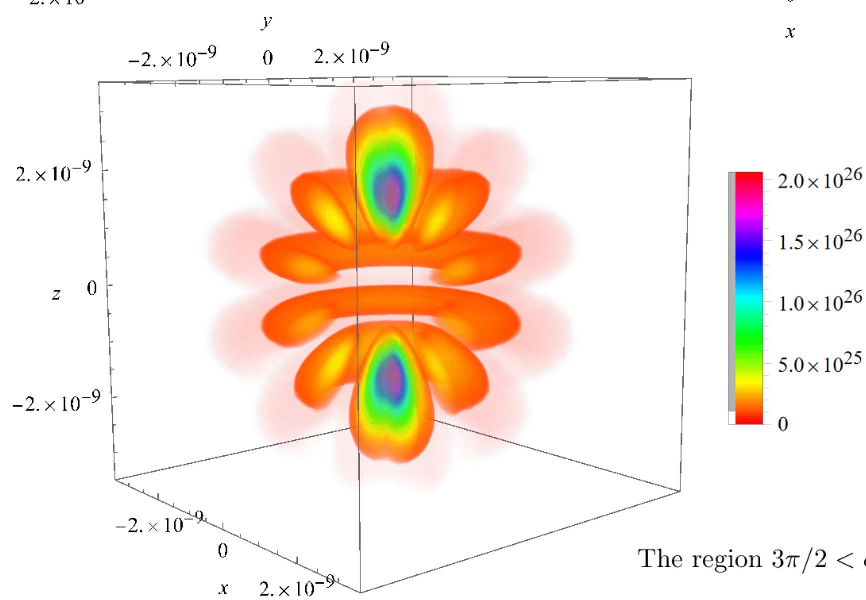


The region $3\pi/2 < \phi < 2\pi$ is excluded.

$$|\Psi_{650}(r, \theta, \phi, t)|^2$$

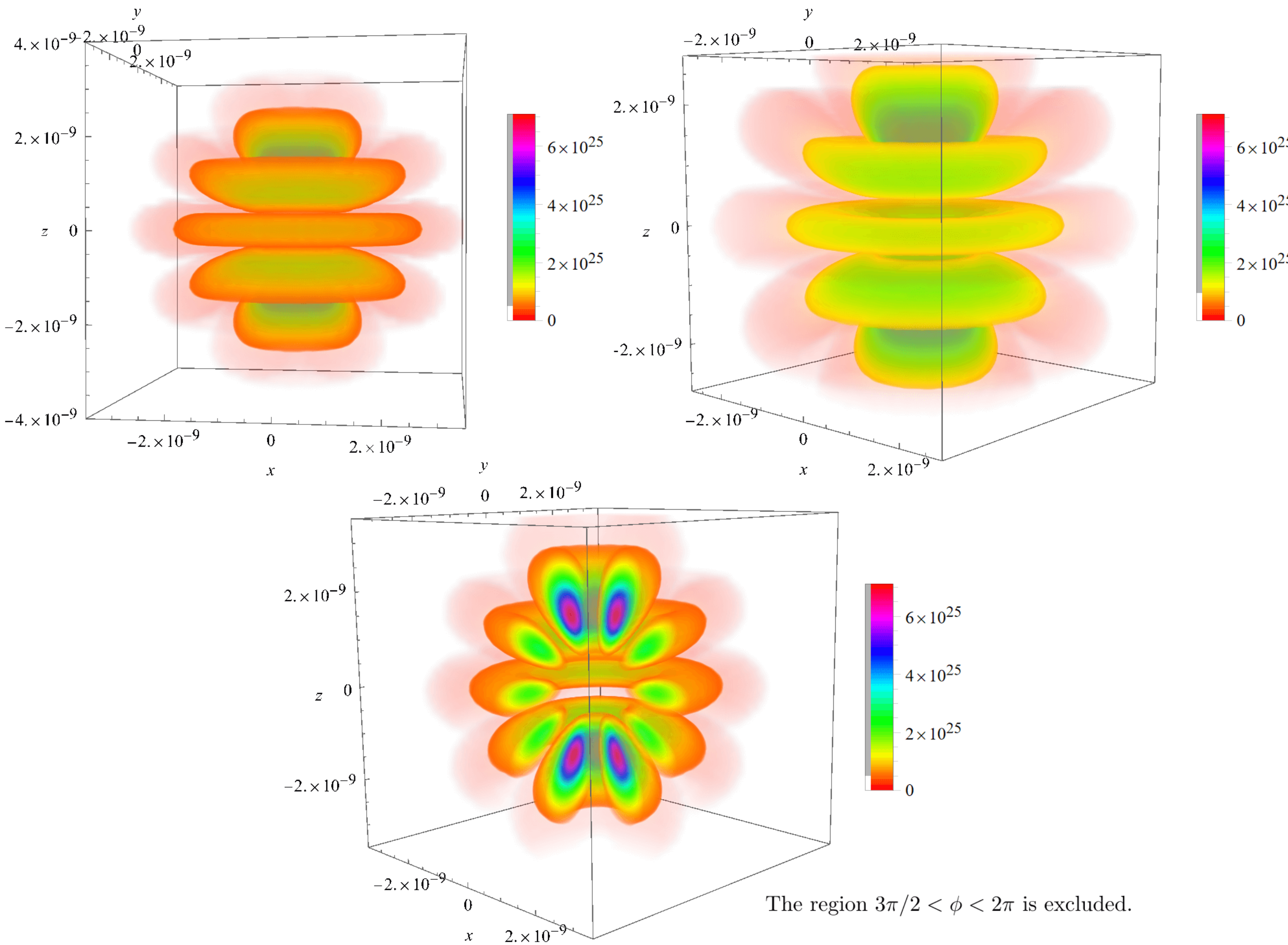






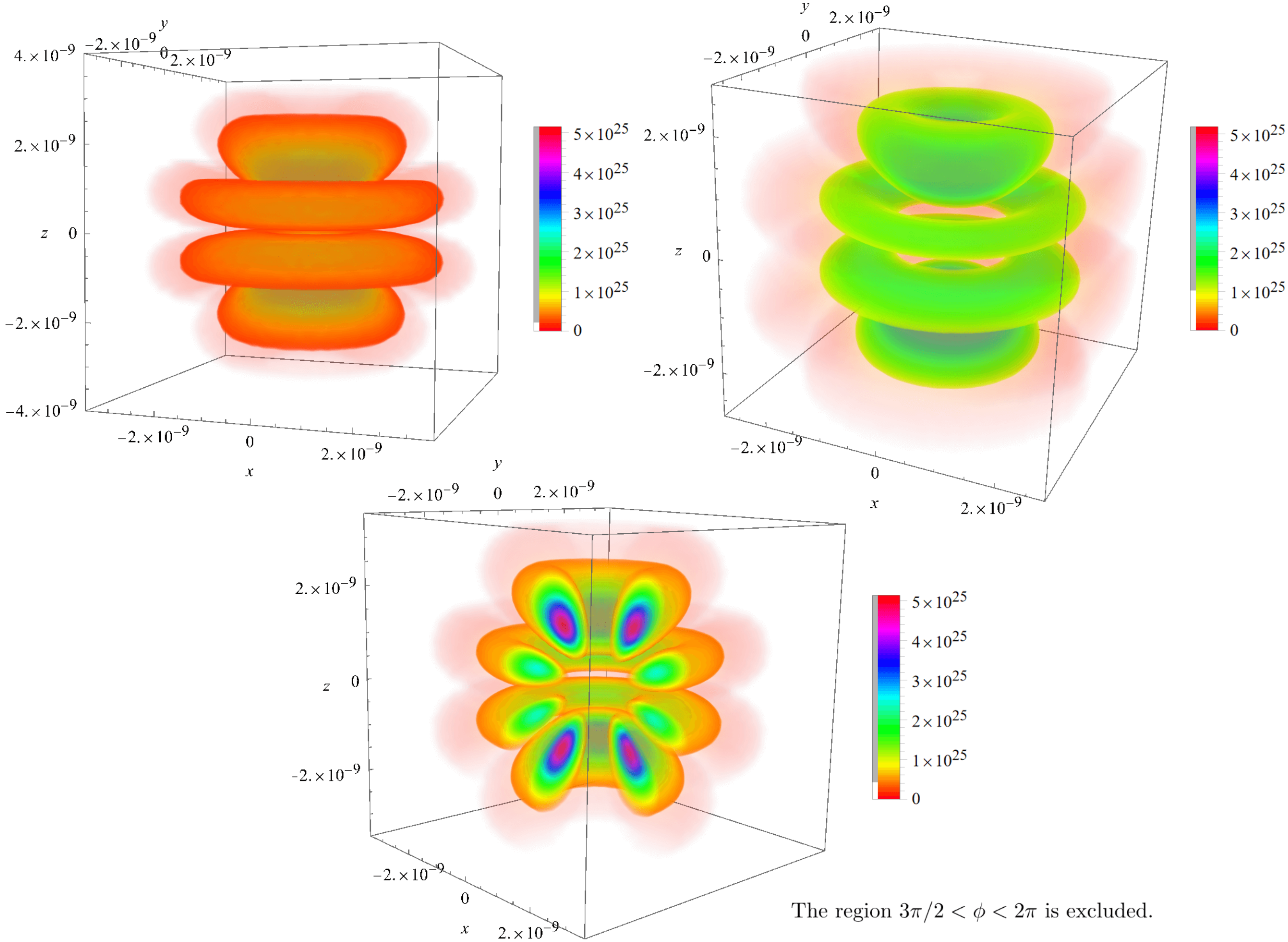
The region $3\pi/2 < \phi < 2\pi$ is excluded.

$$|\Psi_{65\pm1}(r,\theta,\phi,t)|^2$$



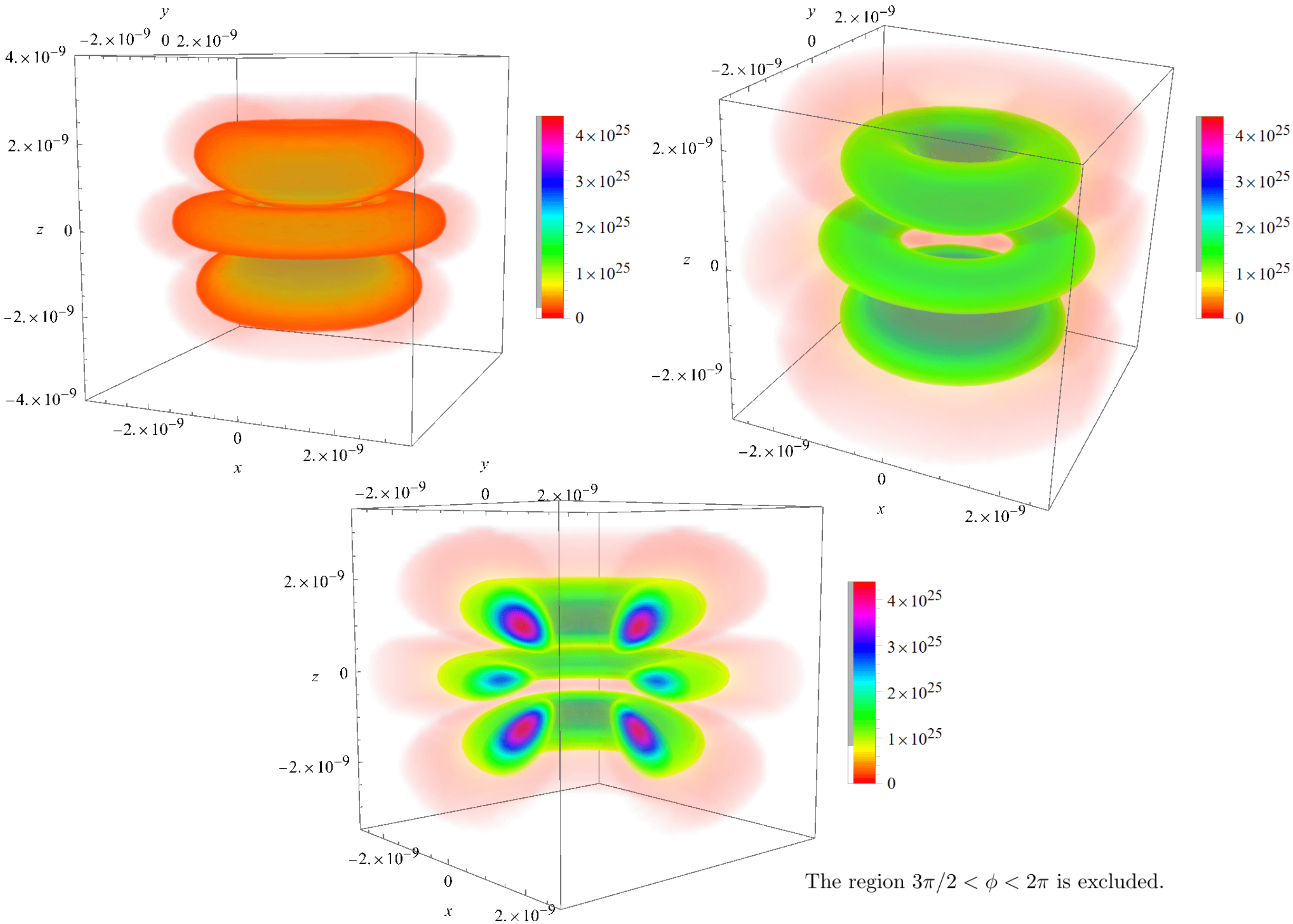
The region $3\pi/2 < \phi < 2\pi$ is excluded.

$$|\Psi_{65\pm2}(r, \theta, \phi, t)|^2$$



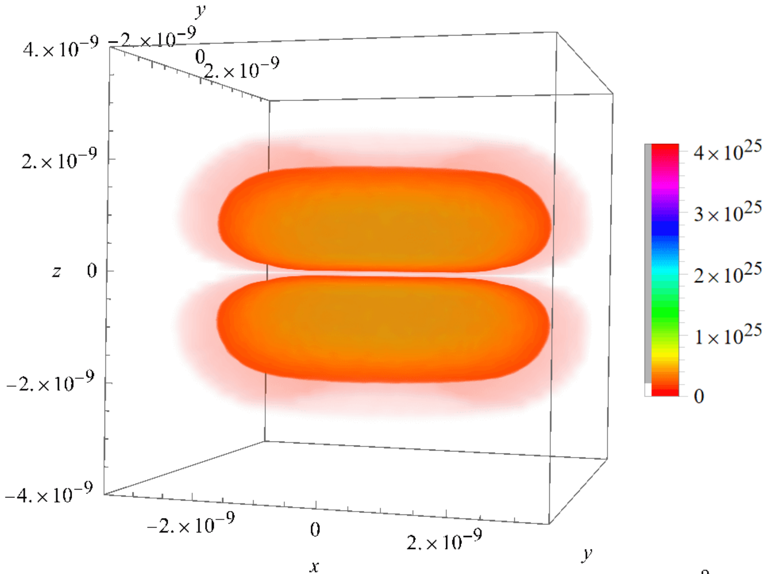
The region $3\pi/2 < \phi < 2\pi$ is excluded.

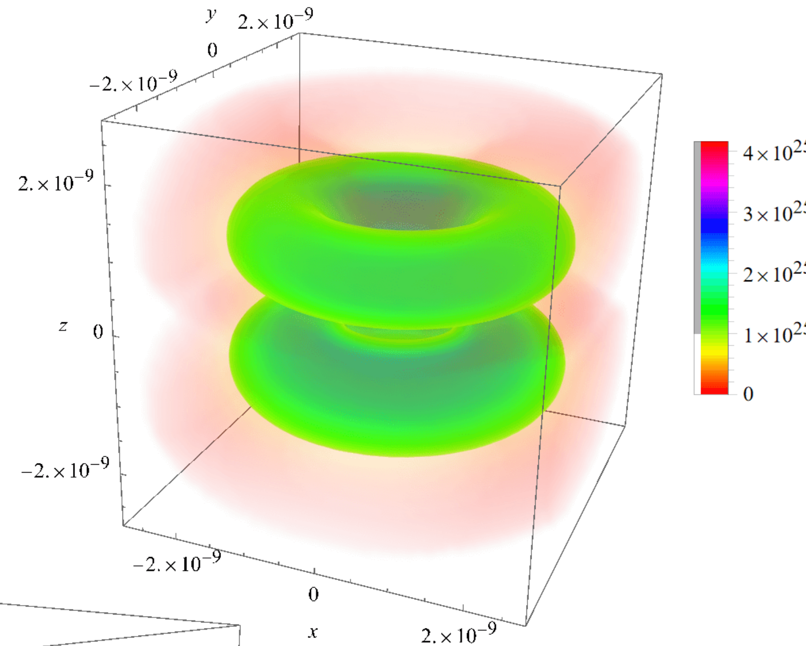
$$|\Psi_{65\pm3}(r, \theta, \phi, t)|^2$$

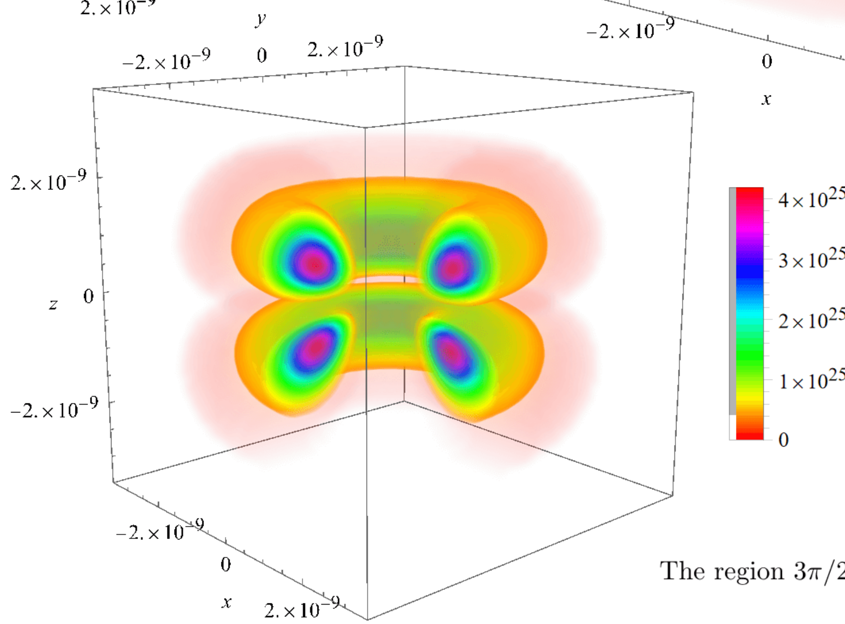


The region $3\pi/2 < \phi < 2\pi$ is excluded.

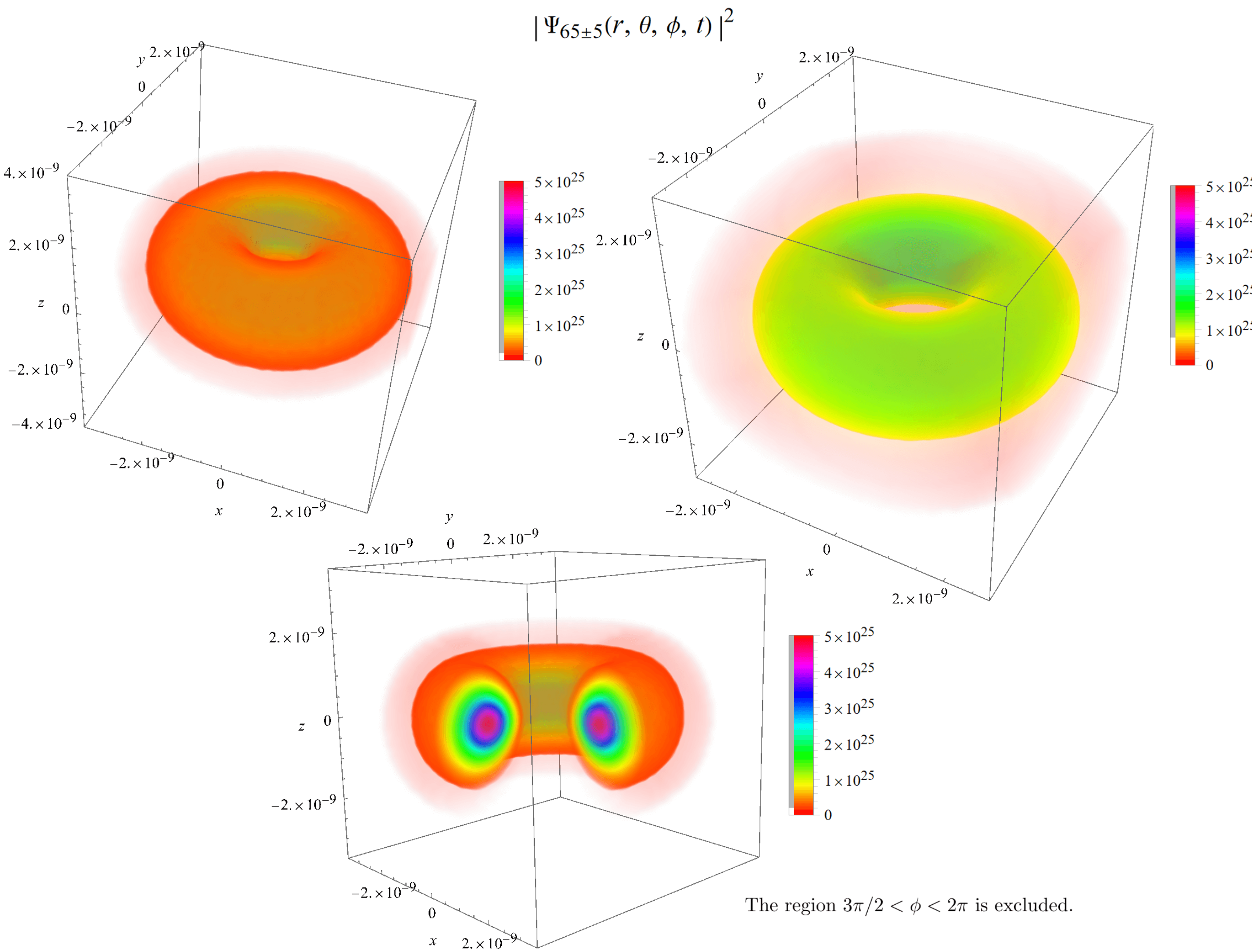
$$|\Psi_{65\pm4}(r,\,\theta,\,\phi,\,t)|^2$$







The region $3\pi/2 < \phi < 2\pi$ is excluded.



The region $3\pi/2 < \phi < 2\pi$ is excluded.